hep-th/0601098
SWAT/06/455

Deconstruction of the Maldacena-Núñez Compactification

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Abstract

We demonstrate a classical equivalence between the large-N limit of the Higgsed $\mathcal{N} = 1^*$ SUSY $U(N)$ Yang-Mills theory and the Maldacena-Núñez twisted compactification of a six dimensional gauge theory on a two-sphere. A direct comparison of the actions and spectra of the two theories reveals them to be identical. We also propose a gauge theory limit which should describe the corresponding spherical compactification of Little String Theory.

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1 Introduction

An interesting development in string theory has been the discovery of consistent interacting non-gravitational quantum theories in dimensions greater than four[1, 2]. A key example arising in six spacetime dimensions is Little String Theory (LST). This theory emerges on the world-volume of NS five-branes in critical string theory and decouples from gravity when the string coupling is taken to zero. The resulting theory is non-local, exhibiting some string-like properties. At low energy Little String Theory reduces to a conventional gauge theory in six-dimensions. This low-energy gauge theory is non-renormalisable and LST is believed to provide a consistent UV completion. These theories are of great interest as they represent an intermediate state between string theories and field theories. A greater understanding of little string theories could improve our understanding of string and gauge theories.

An important challenge in studying these higher-dimensional models is to find a consistent non-perturbative definition of the theory. Deconstruction [3], [4] [1] is a promising approach to this problem which allows one to define higher-dimensional theories as certain limits of better understood four-dimensional gauge theories. In particular, the idea is that a four-dimensional theory in its Higgs phase can be re-interpreted as a theory with extra compact, discretized dimensions. It is hoped that in an appropriate continuum limit the higher-dimensional Lorentz invariance is restored. In recent work [6], one of the authors proposed a new way of deconstructing a toroidally-compactified LST using a marginal deformation of $\mathcal{N} = 4$ SUSY Yang-Mills. One of the advantages of this approach is that the continuum limit can be studied explicitly using S-duality and the AdS/CFT correspondence. In this paper we will present some preliminary evidence that a similar approach using a *relevant* deformation of the $\mathcal{N} = 4$ theory should give a deconstruction of the spherical compactification of LST first studied by Maldacena and Núñez (MN) [7].

The usual starting point for deconstruction is a four-dimensional theory in its Higgs phase. In this case we will study the $\mathcal{N} = 1^*$ SUSY Yang-Mills [8, 9, 10, 11] with gauge group $U(N)$ in a vacuum where the gauge group is broken down to a $U(1)$ subgroup. We will show in detail that, at large-N, this theory is classically equivalent to a twisted compactification of $\mathcal{N} = (1, 1)$ SUSY gauge theory in six dimensions with gauge group $U(1)$.

[1] see also [5]

We demonstrate this equivalence by an explicit comparison of the classical spectra and the Lagrangians of the two theories in question, obtaining exact agreement. A preliminary account of this work, including a comparison of the fermionic spectra, appeared in our earlier letter [12]. In the rest of this introductory section we give a brief overview of these results and explain their potential relevance to the deconstruction of LST.

The $\mathcal{N} = 1^*$ theory contains a $U(N)$ vector multiplet of $\mathcal{N} = 1$ Supersymmetry and three adjoint chiral multiplets of equal mass η. With appropriate rescaling the mass parameter becomes an overall coefficient. Schematically, the superpotential is,

$$\mathcal{W}(\Phi) = \mathrm{Tr}_N \left(i\Phi_1[\Phi_2, \Phi_3] + \frac{1}{2}\sum_{i=1}^{3} \Phi_i^2 \right) \tag{1.1}$$

which leads to the F-flatness condition,

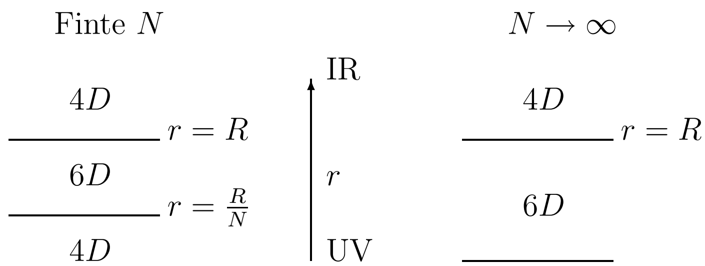
$$[\Phi_i, \Phi_j] = i\varepsilon_{ijk}\Phi_k \tag{1.2}$$

which coincides with the Lie algebra of $SU(2)$. This can be solved by any N-dimensional representation of the $SU(2)$ generators [8]. The same set of vacua are present in both the $U(N)$ and $SU(N)$ cases. By choosing the vacuum $\langle \Phi_i \rangle = J_i^{(N)}$, the N-dimensional representation of the $SU(2)$ generators, we break the gauge group from $U(N)$ to $U(1)$ by the Higgs mechanism. The emergence of the extra dimensions occurs via the mechanism seen in M(atrix) theory [13]. The complex scalars of the chiral multiplets form a fuzzy sphere: a discrete, non-commutative version of the 2-sphere. The expansion of the $\mathcal{N} = 1^*$ fields about this vacuum allows the theory to be interpreted as a six-dimensional non-commutative gauge theory with a UV cutoff in place for finite N. Classically, in the limit $N \to \infty$, we get a commutative, continuum theory on $\Re^{3,1} \times S^2$.

The appearance of a six-dimensional theory can also be understood via the string theory realisation of the $\mathcal{N} = 1^*$ theory [11]. The theory is realised on the worldvolume of N D3 branes in the presence of a background three-form field strength leading to a version of the Myers effect, where the D3 branes polarize into a spherically wrapped D5 brane [14]. The theory living on the worldvolume of the D5 brane reduces to a six-dimensional $U(1)$ gauge theory at low energies. The D3 brane charge is realised as N units of magnetic flux through the 2-sphere leading to non-commutativity in the world-volume gauge theory [15] as expected. The string theory picture also

suggests the identity of the six-dimensional theory. The worldvolume theory on a single D5 brane in flat space is a $U(1)$ gauge theory in six dimensions with $\mathcal{N} = (1,1)$ supersymmetry. The string theory picture suggests that this world-volume theory should be compactified on a 2-sphere in such a way that $\mathcal{N} = 1$ supersymmetry is preserved in the remaining four dimensions. A conventional compactification on a 2-sphere would break all supersymmetries as there are no covariantly constant spinors on S^2. However, in [7], Maldacena and Núñez studied a particular twisted compactification preserving a quarter of the supersymmetries. In their example, the compactfication of the world-volume theory corresponded to wrapping the D5-brane on a non-contractible two-cycle of a Calabi-Yau threefold. In the present case, the brane is wrapped on a topologically trivial cycle supported by the presence of external flux. Our main result is that, despite this difference, the resulting compactified world-volume theory is the same. In particular, as $N \to \infty$, the spectrum of the $\mathcal{N} = 1^*$ theory reproduces that of the six-dimensional theory. Interestingly, the agreement persists at finite N provided we make an appropriate truncation of the six-dimensional Kaluza-Klein spectrum. We also check the agreement directly at the level of the Lagrangian.

The MN compactification and the Higgsed $\mathcal{N} = 1^*$ theory are classically equivalent and are therefore different descriptions of the same classical theory. Which description is most appropriate would depend on the length scale probed. The compact extra dimensions have a scale dictated by the radius of the 2-sphere R which corresponds to inverse mass parameter η^{-1} in the dual gauge theory. At distances $L > R$ the fields have insufficient energy to probe the extra dimensions so the theory appears to be a four-dimensional $\mathcal{N} = 1$ theory with gauge group $U(1)$ and coupling $G_4^2 = g_{ym}^2/N$. At distances $L < R$ the fields have sufficient energy to propagate in the extra dimensions and the theory appears to be the six-dimensional MN compactification with gauge group $U(1)$ and coupling $G_6^2 = 4\pi R^2 g_{ym}^2/N$. As mentioned above the Kaluza-Klein spectrum is truncated at finite N. Correspondingly, at distances $L < R/N$, the theory becomes four-dimensional again and the full $U(N)$ gauge symmetry is restored. Thus the length-scale $R/N \sim 1/(N\eta)$ is the effective UV cutoff or lattice spacing of the six dimensional theory. This behaviour is typical of deconstruction.



An obvious question is whether we can use the four-dimensional gauge theory to define a continuum limit of the six-dimensional theory where the lattice spacing goes to zero. If so we would also like to identify the resulting continuum theory. The situation is closely related to the one described in detail in [6] and we can give a very similar discussion in the present case. To get an interacting theory we must keep the six-dimensional gauge coupling G_6 fixed. It will also be convenient to keep the compactification radius R fixed. The corresponding gauge theory limit involves taking $N \to \infty$ with η and g_{ym}^2/N held fixed. Thus we see that the continuum limit of the six-dimensional theory is a strong coupling limit of the four-dimensional gauge theory. This means we cannot rely on a classical analysis of either theory to study the continuum limit. This again is typical of deconstruction.

However, in the present context, we can make some progress using the exact S-duality of the $\mathcal{N} = 1^*$ theory which takes $g_{ym}^2 \to \tilde{g}_{ym}^2 = 16\pi^2/g_{ym}^2$ and maps the Higgs phase vacuum studied above to a vacuum in a confining phase [9, 10]. The mass parameter η is invariant under S-duality. Under this transformation our proposed continuum limit becomes a 't Hooft large N limit where $N \to \infty$ with $\tilde{g}_{ym}^2 N$ and η are held fixed. As usual, we expect that this limit makes sense and leads to a theory with the qualitative features of a closed string theory. Indeed for $\tilde{g}_{ym}^2 N >> 1$, the corresponding $SU(N)$ theory in this vacuum has a dual description involving a spherically wrapped NS5 brane in an asymptotically AdS geometry [11]. The proposed continuum limit corresponds to a limit of this system where the dual string coupling $g_s = \tilde{g}_{ym}^2/4\pi \to 0$.

In the case of multiple coincident NS-five branes this is precisely the limit used to define LST. Unfortunately, in the present case of a single NS-fivebrane it is not known whether a decoupled world-volume theory exists. However the above results immediately suggest a promising generalisation which involves multiple NS fivebranes. Namely, we should start from an $SU(N)$ theory in the vacuum where the Φ_i take VEVs corresponding to the sum of p irreducible

representations of $SU(2)$ of dimension q;

$$< \Phi_i >= \mathbb{1}_{(p)} \otimes J_i^{(q)}$$

This corresponds to an $SU(N)$ theory with $N = pq$ Higgsed down to $SU(p)$. The low-energy effective theory is $\mathcal{N} = 1$ SUSY Yang-Mills with gauge group $SU(p)$ which confines in the IR at scales of order $\Lambda \sim \eta \exp(-8\pi^2 q/g_{ym}^2 p)$. However, at scales far above Λ, the theory is weakly coupled and we may identify the effective theory by studying the classical action. Indeed a straightforward generalisation of the above results, discussed in Section 5, suggests that the effective theory is a twisted compactification of a six-dimensional $SU(p)$ gauge theory of the Maldacena-Nunez type. Taking a similar continuum limit, $q \to \infty$ with p, g_{ym}^2/q and η held fixed we may perform S-duality to a phase where $U(N)$ is Higgsed to $U(q)$ which is then confined at a lower scale. As before the corresponding limit becomes a 't Hooft-like limit where $N = pq \to \infty$ with $\tilde{g}_{ym}^2 q$, p and η held fixed. The IIB dual now involves p spherically wrapped NS fivebranes [11] in a decoupling limit where $g_s \to 0$. This suggests the $\mathcal{N} = 1^*$ theory really deconstructs the full Little String Theory on the fivebrane worldvolume and not just the low energy six-dimensional gauge theory. More precisely, as in the toroidal case of [6] and in [16], a sub-sector of the four-dimensional theory retains non-vanishing interactions in this limit while the remaining states in the theory decouple. We propose that the interacting sector is equivalent to the MN compactification of LST in this limit. The Little Strings themselves correspond to the confining chromoelectric flux tube of the gauge theory which has fixed tension in the 't Hooft limit.

The rest of the paper is organised as follows. In section 2 we discuss scalars, spinors and vectors on the 2-sphere. In section 3, we determine the Kaluza-Klein spectrum of the MN compactification and the corresponding action for $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory on $\mathcal{R}^{3,1} \times S^2$. In section 4 we describe the emergence of extra dimensions in the Higgsed $\mathcal{N} = 1^*$ theory via deconstruction. In section 5 we review the calculation of the fermionic spectrum of the $\mathcal{N} = 1^*$ theory from [12] in greater detail (for a related calculation see [17]) and calculate the associated bosonic spectrum. We then compare the $\mathcal{N} = 1^*$ spectrum with the Kaluza-Klein spectrum of the MN compactification and show them to be identical in the limit $N \to \infty$. We also discuss briefly the generalisations of our result for other $\mathcal{N} = 1^*$ vacua. In section 6 we will calculate the effective six-dimensional action of the Higgsed $\mathcal{N} = 1^*$ theory. We see that the action is identical to the action of section 3.1, demonstrating the equivalence of the MN compactification and

the Higgsed $\mathcal{N} = 1^*$ theory. In Appendix A we discuss Clifford algebras in various dimensions and review the dimensional reduction of $\mathcal{N} = 1$ SUSY Yang-Mills theory in ten spacetime dimensions to six spacetime dimensions obtaining the $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory.

2 Eigenvalues and Eigenstates on the 2-Sphere

In this section we will discuss fields of different spin on the 2-sphere. To fix our notation and conventions we begin with scalar fields before describing the less well-known subjects of spinors and vectors on the 2-sphere. The 2-sphere is a two-dimensional manifold embedded in $\Re^3$, with a global $SO(3) \sim SU(2)$ isometry group, defined by the equation,

$$x_1^2 + x_2^2 + x_3^2 = R^2 \tag{2.1}$$

for a coordinate basis x_i in $\Re^3$. We define the coordinates x_i in terms of the coordinates on the 2-sphere $q_a = (\theta, \phi)$ and radius R by,

$$x_1 = R \sin\theta \cos\phi \tag{2.2a}$$
$$x_2 = R \sin\theta \sin\phi \tag{2.2b}$$
$$x_3 = R \cos\theta \tag{2.2c}$$

which dictates the metric of the 2-sphere,

$$ds^2 = R^2\, d\theta^2 + R^2 \sin^2\theta\, d\phi^2 \tag{2.3}$$

The generators of $SU(2) \sim SO(3)$ are the angular momentum operators L_i.

$$L_i = -i\varepsilon_{ijk} x_j \partial_k \tag{2.4}$$

In terms of coordinates on the 2-sphere the angular momentum operators are,

$$L_1 = i \sin\phi \frac{\partial}{\partial\theta} + i\cos\phi \cot\theta \frac{\partial}{\partial\phi} \tag{2.5a}$$
$$L_2 = -i\cos\phi \frac{\partial}{\partial\theta} + i\sin\phi\cot\theta\frac{\partial}{\partial\phi} \tag{2.5b}$$
$$L_3 = -i\frac{\partial}{\partial\phi} \tag{2.5c}$$

which we can summarize as [13],

$$L_i = -ik_i^a \partial_a \tag{2.6}$$

The metric tensor can also be expressed in terms of the Killing vectors k_i^a (defined by the above equations) as,

$$g^{ab} = \frac{1}{R^2}\, k_i^a k_i^b \tag{2.7}$$

We can expand any function on the 2-sphere in terms of the eigenfunctions of the 2-sphere,

$$a(\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta,\phi) \tag{2.8}$$

where a_{lm} is a complex coefficient and $Y_{lm}(\theta,\phi)$ are the spherical harmonics, which satisfy the equation,

$$L^2 Y_{lm} = -R^2 \Delta_{S^2} Y_{lm} = l(l+1) Y_{lm} \tag{2.9}$$

where Δ_{S^2} is the scalar Laplacian on the 2-sphere,

$$\Delta_{S^2} = \frac{1}{\sqrt{g}}\, \partial_a (g^{ab} \sqrt{g}\, \partial_b) \tag{2.10}$$

The spherical harmonics have an eigenvalue $\mu \sim l(l+1)$ for integer $l = 0, 1, \ldots,$ with degeneracy $2l+1$. The orthogonality condition of the spherical harmonics is,

$$\int d\Omega\, Y_{lm}^{\dagger} Y_{l'm'} = \delta_{ll'}\, \delta_{mm'} \tag{2.11}$$

where $d\Omega = \sin\theta\, d\theta d\phi$.

The eigenstates of spin-$\frac{1}{2}$ particles on the 2-sphere are the spherical spinors. They are eigenstates of the total angular momentum, $\hat{L}^2$ and $\hat{L}_3$. We form spherical spinors from spherical harmonics and the spin-$\frac{1}{2}$ eigenstates of spin operators S^2 and S_3,

$$\chi_{\frac{1}{2}} = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad \chi_{-\frac{1}{2}} = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.12}$$

The spherical spinors are [18],

$$\Omega_{jlm}(\theta,\phi) = \sum_{\mu} C(l, \frac{1}{2}, j; m-\mu, \mu, m) Y_{l,m-\mu}(\theta,\phi)\chi_{\mu} \tag{2.13}$$

where $C(l, \frac{1}{2}, j; m-\mu, \mu, m)$ are Clebsch-Gordan coefficients. Explicitly the spherical spinors are,

$$\Omega^{\hat{\alpha}}_{l+\frac{1}{2},lm}(\theta,\phi) = \begin{pmatrix} \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}\, Y_{l,m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}\, Y_{l,m+\frac{1}{2}}(\theta,\phi) \end{pmatrix} \tag{2.14}$$

$$\Omega^{\hat{\alpha}}_{l-\frac{1}{2},lm}(\theta,\phi) = \begin{pmatrix} -\sqrt{\frac{l-m+\frac{1}{2}}{2l+1}}\, Y_{l,m-\frac{1}{2}}(\theta,\phi) \\ \sqrt{\frac{l+m+\frac{1}{2}}{2l+1}}\, Y_{l,m+\frac{1}{2}}(\theta,\phi) \end{pmatrix} \tag{2.15}$$

where $\hat{\alpha}$ is a spinor index labelling the two components. The spherical spinors are eigenstates of the Dirac operator on the 2-sphere,

$$\kappa = -\frac{1}{R}\,(1 + \sigma_i L_i) \tag{2.16}$$

Here, the index i on L_i refers to the cartesian basis of the three dimensional Euclidean embedding space of the two sphere. The operator has eigenvalues,

$$\kappa^{\hat{\alpha}}{}_{\hat{\beta}}\, \Omega^{\hat{\beta}}_{q_\pm lm}(\theta,\phi) = \frac{1}{R}\,\kappa_\pm\, \Omega^{\hat{\alpha}}_{q_\pm lm}(\theta,\phi) \tag{2.17}$$

where $\kappa_\pm = \mp(q_\pm + \frac{1}{2})$, $q_\pm = l \pm \frac{1}{2}$. In analogy to the spherical harmonics we can expand a spinor on the 2-sphere in terms of the spherical spinors,

$$\psi^{\hat{\alpha}}(\theta,\phi) = \sum_{l=0}^{\infty} \sum_{m=-q_\pm}^{q_\pm} \left\{ \psi^{(+)}_{lm} \Omega^{\hat{\alpha}}_{l+\frac{1}{2},lm}(\theta,\phi) + \psi^{(-)}_{lm} \Omega^{\hat{\alpha}}_{l-\frac{1}{2},lm}(\theta,\phi) \right\} \tag{2.18}$$

where $\psi^{(\pm)}_{lm}$ are complex coefficients. The orthogonality condition of the spherical spinors is inherited from the spherical harmonics,

$$\int d\Omega\; \Omega^{\dagger}_{qm\hat{\alpha}}(\theta,\phi)\, \Omega^{\hat{\alpha}}_{q'm'}(\theta,\phi) = \delta_{qq'}\delta_{mm'} \tag{2.19}$$

The spherical spinors can also be represented in a spherical basis, in terms of coordinates on the 2-sphere q^a [19], which takes advantage of the local $SO(2)$ symmetry of the 2-sphere. The zweibein (and metric tensor) on the unit 2-sphere are,

$$g_{ab} = \mathrm{diag}(R^2, R^2\sin^2\theta); \qquad e^\alpha_a = \mathrm{diag}(R, R\sin\theta); \qquad g_{ab} = \delta_{\alpha\beta}\, e^\alpha_a\, e^\beta_b \tag{2.20}$$

where we denote the coordinates on the 2-sphere by the index a, b and the local frame by α, β. Derivatives on the 2-sphere are replaced by generally covariant derivatives,

$$\nabla_a \Upsilon = \partial_a \Upsilon + \frac{i}{4}\, R^{\alpha\beta}_a \sigma_{\alpha\beta} \Upsilon \tag{2.21}$$

where $R_a^{\alpha\beta}$ is the spin connection on the 2-sphere and $\sigma_{\alpha\beta}$ are spin-$\frac{1}{2}$ generators of $SO(2)$. The non-zero components of the spin connection on the 2-sphere are,

$$R_\phi^{12} = -R_\phi^{21} = -\cos\theta \tag{2.22}$$

The Clifford algebra in two dimensions is $\hat{\gamma}^a = \{\sigma_1, \sigma_2\}$, where σ_i are the Pauli matrices. We define the generators $\sigma_{\alpha\beta}$ through the Clifford algebra,

$$\sigma_{12} = -\sigma_{21} = -\frac{i}{2}[\hat{\gamma}_1, \hat{\gamma}_2] = \sigma_3 \tag{2.23}$$

The Dirac operator on the 2-sphere is defined as,

$$-i\hat{\nabla}_{S^2} = -ie^{a\alpha}\sigma_\alpha \nabla_a \tag{2.24}$$

Explicitly the Dirac operator is,

$$-i\hat{\nabla}_{S^2} = -\frac{i\sigma_1}{R}\left(\frac{\partial}{\partial\theta} + \frac{\cot\theta}{2}\right) - \frac{i\sigma_2}{R\sin\theta}\frac{\partial}{\partial\phi} \tag{2.25}$$

We calculate the eigenvalues of this Dirac operator by considering the squared Dirac operator,

$$\begin{aligned}(-i\hat{\nabla}_{S^2})^2 = -\frac{1}{R^2}\Big(&\cot\theta\,\partial_\theta + \partial_\theta\partial_\theta + \csc^2\theta\,\partial_\phi\partial_\phi \\ &- i\sigma_3\csc\theta\cot\theta\,\partial_\phi - \frac{1}{4} - \frac{1}{4}\csc^2\theta\Big)\end{aligned} \tag{2.26}$$

The generators of $SU(2)$ in the spin-$\frac{1}{2}$ representation are,

$$\hat{L}_3 = -i\partial_\phi \tag{2.27a}$$

$$\hat{L}_\pm = \pm\, e^{\pm i\phi}\left(\partial_\theta \pm i\cot\theta\,\partial_\phi \pm \frac{1}{2}\sin\theta\,\sigma_3\right) \tag{2.27b}$$

We find the relation between the Casmir operator $\hat{L}^2$ and the square of the Dirac operator is,

$$R^2\,(-i\hat{\nabla}_{S^2})^2 = \hat{L}^2 + \frac{1}{4} \tag{2.28}$$

Therefore the eigenvalues are related and the eigenvalue equation with spherical spinor Υ is,

$$\begin{aligned}\left(-i\hat{\nabla}_{S^2}\right)^2\Upsilon_{jm} &= \frac{1}{R^2}\left(\hat{L}^2 + \frac{1}{4}\right)\Upsilon_{jm} \\ &= \frac{1}{R^2}\left(j(j+1) + \frac{1}{4}\right)\Upsilon_{jm}\end{aligned} \tag{2.29}$$

the eigenvalue is $\mu \sim j(j+1)+\frac{1}{4}$ for $j=\frac{1}{2},\frac{3}{2},\ldots$, with degeneracy $2j+1$. By setting $j=\frac{2l-1}{2}$, $l=1,2,\ldots$ we re-express the eigenvalues in terms of an integer quantum number l, resulting in $\mu \sim l^2$. The allowed eigenvalues correspond to $\mu \sim l^2$ for integer $l \geq 1$ with degeneracy $2l$ [19].

We have two types of spinor, both are a complete orthonormal set of spinors on the 2-sphere. The two types of the spherical spinors, cartesian Ω and spherical Υ are related via a spinor transformation [19]. If we consider two spinors, one in the cartesian basis $\psi(x)$ and one in the spherical basis $\psi(q)$. The transformation is,

$$\psi^{\hat{\alpha}}(x) = (V^{\dagger})^{\hat{\alpha}}{}_{\hat{\beta}}\psi^{\hat{\beta}}(q) \tag{2.30}$$

where the matrix V is,

$$V = \begin{pmatrix} e^{i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) & e^{-i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) \\ -e^{i\frac{\phi}{2}}\sin\left(\frac{\theta}{2}\right) & e^{-i\frac{\phi}{2}}\cos\left(\frac{\theta}{2}\right) \end{pmatrix} \tag{2.31}$$

The Dirac operators are related by the similarity transformation,

$$(\hat{\gamma}_3)^{\hat{\alpha}}{}_{\hat{\gamma}}\,(-i\hat{\nabla}_{S^2})^{\hat{\gamma}}{}_{\hat{\beta}} = V^{\hat{\alpha}}{}_{\hat{\gamma}}\,\kappa^{\hat{\gamma}}{}_{\hat{\delta}}\,(V^{\dagger})^{\hat{\delta}}{}_{\hat{\beta}} \tag{2.32}$$

where $\hat{\gamma}_3 = \sigma_1\sigma_2$ is the two-dimensional chirality operator. The spherical spinors types diagonalize two operators [19], both diagonalize the total angular momentum $\hat{L}_i^2$ (by definition), the spherical spinor Υ diagonalizes the Dirac operator on the 2-sphere $(-i\hat{\nabla}_{S^2})^2$ whilst the spherical spinor Ω diagonalizes the orbital angular momentum L_i^2.

With vector fields on the 2-sphere, it is not consistent to expand the components of the vector separately in spherical harmonics [20], e.g.

$$\begin{aligned} \mathbf{V}(r,\theta,\phi) &= \hat{\mathbf{e}}_r V^r + \hat{\mathbf{e}}_\theta V^\theta + \hat{\mathbf{e}}_\phi V^\phi \\ &= \hat{\mathbf{e}}_r \sum_{jm} v^r_{jm} Y_{jm} + \hat{\mathbf{e}}_\theta \sum_{jm} v^\theta_{jm} Y_{jm} + \hat{\mathbf{e}}_\phi \sum_{jm} v^\phi_{jm} Y_{jm} \end{aligned} \tag{2.33}$$

We must form eigenstates of the total angular momentum for spin-1 particles, the vector harmonics are formed from spherical harmonics and the spin-1 eigenstates of S^2 and S_3,

$$\xi_1 = -\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ i \\ 0 \end{pmatrix} \quad \xi_0 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} \quad \xi_{-1} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -i \\ 0 \end{pmatrix} \tag{2.34}$$

The vector harmonics are [18],

$$\mathbf{Y}_{jlm}(\theta,\phi) = \sum_{\sigma} C(l,1,j;m-\sigma,\sigma,m) Y_{l,m-\sigma}(\theta,\phi)\xi_\sigma \tag{2.35}$$

where j, l, m are integer quantum numbers. We will use a basis which takes advantage of the $SO(2)$ 'Lorentz' symmetry [18, 20],

$$\begin{aligned}
\tilde{\mathbf{T}}_{jm} &= \mathbf{Y}_{jjm}(\theta,\phi) = \frac{1}{\sqrt{j(j+1)}}\left[\frac{\partial Y_{jm}}{\partial\theta}\,\hat{\boldsymbol{\phi}} - \csc\theta\frac{\partial Y_{jm}}{\partial\phi}\,\hat{\boldsymbol{\theta}}\right] && \text{(2.36a)}\\
&= \frac{1}{\sqrt{j(j+1)}}\, i\,\mathbf{L}\, Y_{jm} \\
\tilde{\mathbf{S}}_{jm} &= \sqrt{\frac{j+1}{2j+1}}\mathbf{Y}_{jj-1m}(\theta,\phi) + \sqrt{\frac{j}{2j+1}}\mathbf{Y}_{jj+1m}(\theta,\phi) && \text{(2.36b)}\\
&= \frac{1}{\sqrt{j(j+1)}}\left[\frac{\partial Y_{jm}}{\partial\theta}\,\hat{\boldsymbol{\theta}} + \csc\theta\frac{\partial Y_{jm}}{\partial\phi}\,\hat{\boldsymbol{\phi}}\right] = \frac{1}{\sqrt{j(j+1)}}\,\boldsymbol{\partial}\, Y_{jm} \\
\tilde{\mathbf{R}}_{jm} &= \sqrt{\frac{j}{2j+1}}\mathbf{Y}_{jj-1m}(\theta,\phi) - \sqrt{\frac{j+1}{2j+1}}\mathbf{Y}_{jj+1m}(\theta,\phi) && \text{(2.36c)}\\
&= \hat{\mathbf{r}}\, Y_{jm}
\end{aligned}$$

The harmonics $\tilde{\mathbf{T}}_{jm}$ and $\tilde{\mathbf{S}}_{jm}$ are tangential to the 2-sphere, $\tilde{\mathbf{R}}_{jm}$ is normal to the 2-sphere. Restricting to the vectors on a unit 2-sphere, we set the radial unit vector $\hat{\mathbf{r}} = 0$, therefore $\tilde{\mathbf{R}}_{jm} = 0$. Covariant and contravariant vectors on the 2-sphere can then be defined as,

$$\tilde{V}_i = h_i V^i = h_i^{-1} V_i \tag{2.37}$$

where $g_{ij} = h_i^2 \delta_{ij}$. The corresponding covariant vector harmonics are,

$$\begin{aligned}
\frac{1}{R}\mathbf{T}_{jm} &= \frac{1}{\sqrt{j(j+1)}}\left[\sin\theta\,\partial_\theta Y_{jm}\,\hat{\boldsymbol{\phi}} - \csc\theta\,\partial_\phi Y_{jm}\,\hat{\boldsymbol{\theta}}\right] && \text{(2.38a)}\\
\frac{1}{R}\mathbf{S}_{jm} &= \frac{1}{\sqrt{j(j+1)}}\left[\partial_\theta Y_{jm}\,\hat{\boldsymbol{\theta}} + \partial_\phi Y_{jm}\,\hat{\boldsymbol{\phi}}\right] && \text{(2.38b)}
\end{aligned}$$

The Maxwell field on a 2-sphere is a vector field with the gauge invariance,

$$A_\mu \to A_\mu - R\,\partial_\mu \chi \tag{2.39}$$

The gauge fields can be expanded in vector harmonics and the scalar χ can be expanded in spherical harmonics. Under a gauge transformation the

components transform as,

$$\begin{aligned}
A'_\theta(\theta,\phi) &= R\sum_{jm}\left(t_{jm}\left(-\csc\theta\right)\partial_\phi Y_{jm} + s_{jm}\partial_\theta Y_{jm} - \chi_{jm}\partial_\theta Y_{jm}\right) \\
A'_\phi(\theta,\phi) &= R\sum_{jm}\left(t_{jm}\sin\theta\,\partial_\theta Y_{jm} + s_{jm}\,\partial_\phi Y_{jm} - \chi_{jm}\,\partial_\phi Y_{jm}\right)
\end{aligned}$$

It therefore follows that we can set the complex coefficient $s_{jm} = 0$ via a gauge transformation with $\chi_{jm} = s_{jm}$. The corresponding gauge fixing condition is the generally covariant analogue of the Lorentz Gauge.

$$\begin{aligned}
\nabla^a A_a &= g^{ab}\nabla_b A_a \\
&= g^{ab}\partial_b A_a - g^{ab}\Gamma^c_{ba}A_c
\end{aligned} \tag{2.40}$$

For the 2-sphere there are only three non-zero Christoffel symbols,

$$\Gamma^\theta_{\phi\phi} = -\cos\theta\sin\theta \qquad \Gamma^\phi_{\theta\phi} = \Gamma^\phi_{\phi\theta} = \cot\theta \tag{2.41}$$

Therefore,

$$\begin{aligned}
\nabla^a A_a &= g^{ab}\partial_b A_a + \frac{1}{R^2}\cot\theta\,A_\theta \\
&= \frac{1}{R}\sum_{jm}\frac{1}{\sqrt{j(j+1)}}\Big\{t_{jm}\Big(\partial_\theta(-\csc\theta\,\partial_\phi Y_{jm}) - \cot\theta\csc\theta\,\partial_\phi Y_{jm} \\
&\quad + \csc\theta\,\partial_\phi\partial_\theta Y_{jm}\Big) + s_{jm}\Big(\partial_\theta\partial_\theta Y_{jm} + \cot\theta\,\partial_\theta Y_{jm} + \csc^2\theta\,\partial_\phi\partial_\phi Y_{jm}\Big)\Big\} \\
&= -R\sum_{jm}\frac{1}{\sqrt{j(j+1)}}\,s_{jm}\Delta_{S^2}Y_{jm}
\end{aligned} \tag{2.42}$$

The gauge condition $\nabla^a A_a = 0$ thus sets $s_{jm} = 0$. The orthonormality condition for the vector harmonic $\mathbf{T}_{lm}$ is,

$$\int d\Omega\,\mathbf{T}^\dagger_{lm}\mathbf{T}_{l'm'} = \delta_{ll'}\delta_{mm'} \tag{2.43}$$

And it is an eigenfunction of the total angular momentum $\hat{L}^2$ and orbital angular momentum L^2,

$$L^2\mathbf{T}_{lm} = l(l+1)\mathbf{T}_{lm} \tag{2.44}$$

It follows that we can expand any gauge field on the 2-sphere as,

$$\mathbf{A} = \sum_{l=1}^{\infty}\sum_{m=-l}^{l} a_{lm}\mathbf{T_{lm}} \tag{2.45}$$

3 Twisted Compactification

In this Section we study the Maldacena-Núñez compactification of $\mathcal{N} = (1,1)$ SUSY Yang-Mills in six dimensions and its classical spectrum of Kaluza-Klein modes as in [12]. We start from the $U(1)$ theory defined on a six-dimensional Minkowski space $\Re^{5,1}$. The global symmetry group is

$$SO(5,1) \times SO(4) \simeq SU(4) \times SU(2)_A \times SU(2)_B \tag{3.1}$$

Here $SO(5,1)$ is the six-dimensional Lorentz group and $SO(4)$ is the R-symmetry of the $\mathcal{N} = (1,1)$ superalgebra. The matter content is a single $U(1)$ vector multiplet of $\mathcal{N} = (1,1)$ supersymmetry. It contains [2] a six-dimensional gauge field A_M, four real scalar fields ϕ_i and two Weyl spinors of opposite chirality; λ_l and $\tilde{\lambda}_{\bar{l}}$. Their transformation properties under the global symmetries are,

	$SU(4)$	$SU(2)_A$	$SU(2)_B$
A_M	$\mathbf{6}$	$\mathbf{1}$	$\mathbf{1}$
ϕ_i	$\mathbf{1}$	$\mathbf{2}$	$\mathbf{2}$
λ_l	$\mathbf{4}$	$\mathbf{2}$	$\mathbf{1}$
$\tilde{\lambda}_{\bar{l}}$	$\bar{\mathbf{4}}$	$\mathbf{1}$	$\mathbf{2}$

Anticipating compactification of two spatial dimensions, we write the space-time as $\Re^{5,1} \sim \Re^{3,1} \times \Re^2$. This decomposition breaks the six-dimensional Lorentz group down to a subgroup,

$$H = SO(3,1) \times SO(2) \tag{3.2}$$

with covering group,

$$\bar{H} = SU(2)_L \times SU(2)_R \times U(1)_{45} \tag{3.3}$$

It is straightforward to decompose the six-dimensional fields into representations of H. The gauge field is written as,

$$\begin{aligned} A_M &= A_\mu \qquad \mu = 0,1,2,3 \\ &= A_a \qquad a = M-3 = 4,5 \end{aligned} \tag{3.4}$$

and we define the complex fields (taking $SO(2) \to U(1)$),

$$n_\pm = \frac{1}{\sqrt{2}} (A_4 \pm i A_5) \tag{3.5}$$

[2] The corresponding spacetime indices run over $M = 0,1,\ldots,5$, $i = 1,\ldots,4$, $l = 1,\ldots,4$, $\bar{l} = 1,\ldots,4$

Under the decomposition of the $SU(4)$ covering group, the six-dimensional spinors λ_l and $\tilde{\lambda}_{\bar{l}}$, transforming in the $\mathbf{4}$ and $\bar{\mathbf{4}}$, split according to,

$$\begin{aligned} \mathbf{4} &\rightarrow (\mathbf{2},\mathbf{1})^{+1} \oplus (\mathbf{1},\mathbf{2})^{-1} \\ \bar{\mathbf{4}} &\rightarrow (\mathbf{2},\mathbf{1})^{-1} \oplus (\mathbf{1},\mathbf{2})^{+1} \end{aligned}$$

under $SU(2)_L \times SU(2)_R$ where the superscript denotes $U(1)_{45}$ charge. Thus we obtain a total of four left-handed Weyl spinors $\lambda^{\alpha}_{\underline{\alpha}}$, $\psi^{\alpha}_{\underline{\dot{\alpha}}}$ and four right-handed spinors $\bar{\lambda}^{\dot{\alpha}}_{\underline{\alpha}}$, $\bar{\psi}^{\dot{\alpha}}_{\underline{\dot{\alpha}}}$. Here α and $\dot{\alpha}$ are the usual $SU(2)_L$ and $SU(2)_R$ indices; $\underline{\alpha}$ and $\underline{\dot{\alpha}}$ are indices of $SU(2)_A$ and $SU(2)_B$ respectively.

To summarize, the resulting bosonic fields then have quantum numbers,

	$SU(2)_L$	$SU(2)_R$	$U(1)_{45}$	$SU(2)_A$	$SU(2)_B$
A_μ	$\mathbf{2}$	$\mathbf{2}$	0	$\mathbf{1}$	$\mathbf{1}$
$n_\pm$	$\mathbf{1}$	$\mathbf{1}$	± 2	$\mathbf{1}$	$\mathbf{1}$
ϕ_i	$\mathbf{1}$	$\mathbf{1}$	0	$\mathbf{2}$	$\mathbf{2}$

while the fermions transform as,

	$SU(2)_L$	$SU(2)_R$	$U(1)_{45}$	$SU(2)_A$	$SU(2)_B$
$\lambda^{\alpha}_{\underline{\alpha}}$	$\mathbf{2}$	$\mathbf{1}$	$+1$	$\mathbf{2}$	$\mathbf{1}$
$\bar{\lambda}^{\dot{\alpha}}_{\underline{\alpha}}$	$\mathbf{1}$	$\mathbf{2}$	-1	$\mathbf{2}$	$\mathbf{1}$
$\psi^{\alpha}_{\underline{\dot{\alpha}}}$	$\mathbf{2}$	$\mathbf{1}$	-1	$\mathbf{1}$	$\mathbf{2}$
$\bar{\psi}^{\dot{\alpha}}_{\underline{\dot{\alpha}}}$	$\mathbf{1}$	$\mathbf{2}$	$+1$	$\mathbf{1}$	$\mathbf{2}$

We now compactify the theory by replacing the 45-plane by a sphere. In conventional compactification the couplings of fields to the curvature of the sphere are determined by their quantum numbers under $U(1)_{45}$ which corresponds to local rotations in the 45-plane. In the compactification of Maldacena and Núñez the theory is twisted by embedding the local rotation group into the $SU(2)_A \times SU(2)_B$ R-symmetry group of the theory. To accomplish this we define Cartan subgroups $U(1)_A$ and $U(1)_B$ of $SU(2)_A$ and $SU(2)_B$ with corresponding generators Q_A and Q_B respectively[3]. We also define the diagonal subgroup $U(1)_T = D(U(1)_{45} \times U(1)_A)$ with generator $Q_T = Q_{45} + Q_A$. The vector multiplet fields then have quantum numbers,

[3]These generators are normalised to take the values $Q_A = \pm 1$ on states in the fundamental representation of $SU(2)$.

	$U(1)_A$	$U(1)_T$
A_μ	0	0
$n_\pm$	0	± 2
ϕ_i	± 1	± 1
$\lambda^\alpha_{\underline{\alpha}}$	± 1	$\begin{pmatrix} +2 \\ 0 \end{pmatrix}$
$\bar\lambda^{\dot\alpha}_{\underline{\alpha}}$	± 1	$\begin{pmatrix} 0 \\ -2 \end{pmatrix}$
$\psi^\alpha_{\underline{\dot\alpha}}$	0	-1
$\bar\psi^{\dot\alpha}_{\underline{\dot\alpha}}$	0	$+1$

We then compactify the theory with $U(1)_T$ playing the role of the local rotation group (instead of $U(1)_{45}$). We will refer to the $U(1)_T$ quantum number as T-spin and the six-dimensional fields can be split up accordingly as,

$$
\begin{array}{lll}
\text{T-scalars:} & Q_T = 0 & A_\mu,\ \lambda^\alpha_{\underline{\alpha}=2},\ \bar\lambda^{\dot\alpha}_{\underline{\alpha}=1} \\
\text{T-spinors:} & Q_T = \pm 1 & \psi^\alpha_{\underline{\dot\alpha}},\ \bar\psi^{\dot\alpha}_{\underline{\dot\alpha}},\ \phi_i \\
\text{T-vectors:} & Q_T = \pm 2 & n_\pm,\ \lambda^\alpha_{\underline{\alpha}=1},\ \bar\lambda^{\dot\alpha}_{\underline{\alpha}=2}
\end{array}
$$

Correspondingly the terms scalar, spinor and vector will be reserved for describing the transformation properties of fields under the four-dimensional Lorentz group. The existence of a single Weyl spinor (of both chiralities) which is also a T-scalar guarentees the existence of a single massless fermion in four-dimensions as required by $\mathcal{N} = 1$ supersymmetry.

3.1 Maldacena-Núñez Bosonic Action

A direct calculation of the MN action can be performed by the twisted compactification of the $\mathcal{N} = (1,1)$ SUSY Yang-Mills action, allowing a comparison with the effective six-dimensional action of the Higgsed $\mathcal{N} = 1^*$ theory. In this section we will calculate the bosonic part of this action. The $\mathcal{N} = (1,1)$ action with gauge group $U(1)$ is obtained via the trivial dimensional reduction of the $\mathcal{N} = 1$ SUSY Yang-Mills theory in ten spacetime dimensions with gauge group $U(1)$ (see Appendix A for more details). The action for this $U(1)$ theory is,

$$\mathcal{S} = \frac{1}{g_6^2} \int d^6x \left\{ -\frac{1}{4} F_{ij}F^{ij} - \frac{1}{2} \partial_i \phi_n \partial^i \phi^n \right\} \tag{3.6}$$

where $i,j = 0,1,\ldots,5$ and $n = 1,\ldots,4$. We split the manifold $\Re^{5,1} \rightarrow \Re^{3,1} \times \Re^2$, and let $\mu = 0,1,2,3$ and $a = 1,2$.

$$
\begin{aligned}
\mathcal{S} = \frac{1}{g_6^2} \int d^6x \Big\{ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}F_{\mu a}F^{\mu a} - \frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}\partial_\mu\phi_m\partial^\mu\phi^m \\
& -\frac{1}{2}\partial_a\phi_m\partial^a\phi^m \Big\} \\
= \frac{1}{g_6^2} \int d^6x \Big\{ & -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{2}\partial_\mu n_a\partial^\mu n^a - \partial_\mu n_a \partial^a A^\mu - \frac{1}{2}\partial_a A_\mu \partial^a A^\mu \\
& -\frac{1}{4}F_{ab}F^{ab} - \frac{1}{2}\partial_\mu\phi_m\partial^\mu\phi^m - \frac{1}{2}\partial_a\phi_m\partial^a\phi^m \Big\}
\end{aligned}
$$

We want to perform the twisted compactification on the sphere S^2. $U(1)_T \sim SO(2)_T$ is the local rotation group in the twisted compactification. From the group structure in section 3 we see that A_μ is a T-scalar, ϕ_m form T-spinors and n_a form T-vectors. In moving from a flat spacetime to a curved spacetime, derivatives on the flat spacetime become general covariant derivatives. $\partial_\mu \rightarrow \partial_\mu$ and $\partial_a \rightarrow \nabla_a$.

$$F_{\mu\nu} \rightarrow F_{\mu\nu} \tag{3.7a}$$

$$\partial_\mu n_a \rightarrow \partial_\mu n_a \tag{3.7b}$$

A general covariant derivative's action on a scalar is that of an ordinary derivative.

$$\partial_a A_\mu \rightarrow \nabla_a A_\mu = \partial_a A_\mu \tag{3.8}$$

For a vector $\partial_a n_b \rightarrow \partial_a n_b - \Gamma^c_{ab} n_c$, where Γ^c_{ab} is a Christoffel symbol.

$$
\begin{aligned}
F_{ab} = \partial_a n_b - \partial_b n_a \rightarrow \mathcal{F}_{ab} &= \partial_a n_b - \Gamma^c_{ab} n_c - \partial_b n_a + \Gamma^c_{ba} n_c \\
&= \partial_a n_b - \partial_b n_a
\end{aligned} \tag{3.9}
$$

Consider the real scalars ϕ_m of $SO(4)$. In order to transform these real scalars into T-spinors on the sphere we need to reveal the spin structure of $SU(2)_A$. We construct an $SO(4)$ bispinor,

$$
\begin{aligned}
v_{\underline{\alpha\dot{\alpha}}} &= i(\tau^m)_{\underline{\alpha\dot{\alpha}}}\phi_m \\
\phi^m &= -\frac{i}{2}(\bar{\tau}^m)^{\underline{\dot{\alpha}\alpha}} v_{\underline{\alpha\dot{\alpha}}}
\end{aligned}
$$

Substituting this expression for the real scalars into the last term of the

action,

$$\begin{aligned}\int d^2x\,\partial_a\phi_m\partial^a\phi^m &= -\frac{1}{4}\int d^2x\,\partial_a\left((\bar{\tau}^m)^{\dot{\alpha}\underline{\alpha}}v_{\underline{\alpha}\dot{\alpha}}\right)\partial^a\left((\bar{\tau}_m)^{\dot{\beta}\underline{\beta}}v_{\underline{\beta}\dot{\beta}}\right)\\ &= \frac{1}{2}\int d^2x\,\partial_a v^{\underline{\alpha}\dot{\alpha}}\partial^a v_{\underline{\alpha}\dot{\alpha}}\\ &= \frac{1}{2}\int d^2x\,v^{\underline{\alpha}}{}_{\dot{\alpha}}\partial_a\partial^a v_{\underline{\alpha}}{}^{\dot{\alpha}}\\ &= -\frac{1}{2}\int d^2x\,\Xi^\dagger_{\hat{\alpha}}(\partial_a\partial^a\otimes\mathbb{1}_2)^{\hat{\alpha}}{}_{\hat{\beta}}\,\Xi^{\hat{\beta}}\end{aligned}$$

where $\Xi^{\hat{\alpha}} = \begin{pmatrix} v_{\underline{1}}{}^{\dot{\alpha}} \\ v_{\underline{2}}{}^{\dot{\alpha}} \end{pmatrix}$ and $(v_{\underline{\alpha}}{}^{\dot{\alpha}})^\dagger = (\lambda^A_{\underline{\alpha}}\bar{\lambda}^{\dot{\alpha}}_A)^\dagger = -v^{\underline{\alpha}}{}_{\dot{\alpha}}$. The differential operator can be rewritten,

$$\partial_a\partial^a\otimes\mathbb{1}_2 = \sigma^a\partial_a\sigma^b\partial_b = \not{\partial}^2 \tag{3.10}$$

Moving from flat spacetime to curved spacetime, $\not{\partial}\to\hat{\nabla}_{S^2}$,

$$-\frac{1}{2}\,\Xi^\dagger_{\hat{\alpha}}(\not{\partial}^2)^{\hat{\alpha}}{}_{\hat{\beta}}\Xi^{\hat{\beta}} \;\to\; -\frac{1}{2}\,\Xi^\dagger_{\hat{\alpha}}(\hat{\nabla}^2_{S^2})^{\hat{\alpha}}{}_{\hat{\beta}}\,\Xi^{\hat{\beta}} = \frac{1}{2}\Xi^\dagger_{\hat{\alpha}}[(-i\hat{\nabla}_{S^2})^2]^{\hat{\alpha}}{}_{\hat{\beta}}\,\Xi^{\hat{\beta}} \tag{3.11}$$

Similarly,

$$\partial_\mu\phi_m\partial^\mu\phi^m = \frac{1}{2}\partial_\mu\Xi^\dagger_{\hat{\alpha}}\partial^\mu\Xi^{\hat{\alpha}} \tag{3.12}$$

Therefore the bosonic MN action is,

$$\begin{aligned}\mathcal{S}_B = \frac{1}{g_6^2}\int d^4x\int R^2 d\Omega\,\Big\{&-\frac{1}{4}\,F_{\mu\nu}F^{\mu\nu}-\frac{1}{2}\,\partial_\mu n_a\partial^\mu n^a-\partial_\mu n_a\partial^a A^\mu\\ &-\frac{1}{2}\,\partial_a A_\mu\partial^a A^\mu-\frac{1}{4}\,\mathcal{F}_{ab}\mathcal{F}^{ab}-\frac{1}{4}\,\partial_\mu\Xi^\dagger_{\hat{\alpha}}\partial^\mu\Xi^{\hat{\alpha}}-\frac{1}{4}\,\Xi^\dagger_{\hat{\alpha}}[(-i\hat{\nabla}_{S^2})^2]^{\hat{\alpha}}{}_{\hat{\beta}}\,\Xi^{\hat{\beta}}\Big\}\end{aligned} \tag{3.13}$$

3.2 Maldacena-Núñez Kaluza-Klein Spectrum

Each six-dimensional field has a kinetic term on S^2. After expanding in appropriate spherical harmonics, this kinetic term determines the masses of an infinite tower of four-dimensional fields. We now consider the Kaluza-Klein spectrum of each type of field in turn.

After integration by parts, the kinetic term for a T-scalar field A_μ defined on a 2-sphere of unit radius can be written as,

$$\mathcal{S}_A = \frac{1}{g_6^2} \int R^2 d\Omega \; A_\mu \Delta_{S^2} A^\mu \tag{3.14}$$

To find the mass eigenstates we expand A_μ in terms of spherical harmonics as,

$$A_\mu(\theta, \phi) \;\; = \;\; \sum_{l=0}^{\infty} \sum_{m=-l}^{+l} A_{(\mu)lm} Y_{lm}(\theta, \phi) \tag{3.15}$$

From section 2 we know the eigenvalues of the scalar Laplacian are $\mu \sim l(l+1)$ for integer $l \geq 0$ and for $m = -l, \ldots + l$. Thus for each T-scalar field in six dimensions we find a Kaluza-Klein tower of four-dimensional fields with masses,

$$M^2 = \frac{1}{R^2} l(l+1) \tag{3.16}$$

with degeneracy $(2l+1)$. According to the list of T-scalar fields given above we find a four-dimensional vector field, a left-handed Weyl spinor and a right-handed Weyl spinor at each mass level. The corresponding representations of the four-dimensional Lorentz group $SU(2)_L \times SU(2)_R$ are,

$$(\mathbf{2}, \mathbf{2}) \oplus (\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2})$$

For $l = 0$, the corresponding four-dimensional fields are massless and the spin quantum numbers match those of a single massless vector multiplet of $\mathcal{N} = 1$ supersymmetry. For $l > 1$, we find massive vector fields together with Weyl fermions. However, a massive vector multiplet of $\mathcal{N} = 1$ supersymmetry also includes scalar fields in four-dimensions. Thus, for $l > 0$, the fields descending from the T-scalars in six dimensions do not form complete multiplets of $\mathcal{N} = 1$ supersymmetry. This puzzle will be resolved below where we will find the additional states needed to form massive vector multiplets.

A two-component Dirac T-spinor Υ defined on a 2-sphere has kinetic term,

$$\mathcal{S}_\Upsilon = \frac{1}{g_6^2} \int R^2 d\Omega \; i\bar{\Upsilon} \, \hat{\nabla}_{S^2} \Upsilon \tag{3.17}$$

From section 2 we know the eigenvalues of the Dirac operator on the 2-sphere correspond to $\mu \sim l^2$ for integer $l \geq 1$ with degeneracy $2l$. The fermionic T-spinor fields, $\psi^{\alpha}_{\underline{\dot{\alpha}}}$ and $\bar{\psi}^{\dot{\alpha}}_{\underline{\dot{\alpha}}}$ listed above can be combined to form two-component

Dirac spinors on S^2 with kinetic terms (3.17) according to,

$$\Upsilon^{\alpha}_{(\underline{\dot{\alpha}})} = \left(\begin{array}{c} \psi^{\alpha}_{\underline{\dot{\alpha}}} \\ \bar{\psi}^{\dot{\alpha}=\alpha}_{\underline{\dot{\alpha}}} \end{array} \right) \tag{3.18}$$

for $\underline{\dot{\alpha}} = 1, 2$, $\alpha = 1, 2$. Thus we obtain four-species of Dirac spinors on the 2-sphere. Each species yields $2l$ states of mass,

$$M^2 = \frac{1}{R^2} l^2 \tag{3.19}$$

for $l \geq 1$ after expansion in terms of eigenstates of the squared Dirac operator. At each mass level we therefore find $8l$ off-shell degrees of freedom which can be recombined as $4l$ left-handed and $4l$ right-handed Weyl spinors in four dimensions. These Weyl spinors must be paired with bosonic fields to form multiplets of $\mathcal{N} = 1$ SUSY in four dimensions. The extra fields come from Kaluza-Klein reduction of the bosonic T-spinors ϕ_i, $i = 1, 2, 3, 4$, which yield massive scalar fields in four dimensions. These states combine with the fermionic T-spinors to form massive chiral multiplets with Lorentz spins,

$$(\mathbf{2}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{2}) \oplus 2 \times (\mathbf{1}, \mathbf{1})$$

It remains to determine the Kaluza-Klein spectrum of the T-vector fields. As we have unbroken $\mathcal{N} = 1$ supersymmetry in the four non-compact dimensions it suffices to focus on the bosonic T-vector fields $n_{\pm} = (A_4 \pm iA_5)/\sqrt{2}$. The two real components A_4 and A_5 define a Maxwell gauge field n_a on the 2-sphere. The resulting kinetic term reads,

$$\mathcal{S}_n = \frac{1}{g_6^2} \int R^2 d\Omega \; \frac{1}{4} \mathcal{F}_{ab} \mathcal{F}^{ab} \tag{3.20}$$

where $\mathcal{F}_{ab} = \partial_a n_b - \partial_b n_a$. We impose the gauge condition $\nabla^a n_a = 0$ and expand $\mathbf{n} = (n_\theta, n_\phi)$ in terms of vector spherical harmonics,

$$\mathbf{n} \;\; = \;\; \sum_{l,m} n_{lm} \mathbf{T_{lm}}$$

The field tensor expands as,

$$\begin{aligned} \mathcal{F}_{\theta\phi} &= R \sum_{l,m} n_{lm} \frac{1}{\sqrt{l(l+1)}} \Big(\partial_\theta (\sin\theta \, \partial_\theta Y_{lm}) + \csc\theta \, \partial_\phi \partial_\phi Y_{lm} \Big) \\ &= R \sum_{l,m} n_{lm} \frac{1}{\sqrt{l(l+1)}} \sin\theta \, L^2 Y_{lm} \end{aligned} \tag{3.21}$$

The Maxwell term is,

$$\mathcal{S}_n = \frac{1}{g_6^2} \sum_{l,m,l',m'} n^\dagger_{lm} n_{l'm'}\, l(l+1)\, \delta_{ll'}\delta_{mm'} \tag{3.22}$$

Thus the T-vector field $n_\pm$ yields a Kaluza-Klein tower of four-dimensional scalar fields of mass,

$$M^2 = \frac{1}{R^2}\, l(l+1) \tag{3.23}$$

with degeneracy $2l+1$ for $l \geq 1$. Notice that these fields are degenerate in mass with the four-dimensional vector fields coming from the Kaluza-Klein reduction of the T-scalars. In fact the number of scalar fields is just right to pair up with the massive vector fields to form massive vector multiplets of $\mathcal{N}=1$ SUSY in four dimensions with Lorentz spins,

$$(\mathbf{2},\mathbf{2}) \oplus 2 \times [(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})] \oplus (\mathbf{1},\mathbf{1})$$

The fermionic part of each multiplet includes two species of left- and right-handed Weyl fermions. One species comes from the KK reduction of the fermionic T-scalars $\lambda^\alpha_{\underline{\alpha}=2}$ and $\bar{\lambda}^{\dot{\alpha}}_{\underline{\alpha}=1}$ and the other comes from the reduction of the fermionic T-vectors $\lambda^\alpha_{\underline{\alpha}=1}$ and $\bar{\lambda}^{\dot{\alpha}}_{\underline{\alpha}=2}$.

We summarise the complete Kaluza-Klein spectrum of the MN compactification in the table below,

T-scalar:

λ	States
$l(l+1)$	$(2l+1)\times \left\{(\mathbf{2},\mathbf{2}) \oplus (\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2})\right\}$

$l = 0, 1, 2, \ldots$

T-spinor:

λ	States
l^2	$4l\times \left\{(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \oplus 2\times(\mathbf{1},\mathbf{1})\right\}$

$l = 1, 2, 3, \ldots$

T-vector:

λ	States
$l(l+1)$	$(2l+1)\times \left\{(\mathbf{2},\mathbf{1}) \oplus (\mathbf{1},\mathbf{2}) \oplus 2\times(\mathbf{1},\mathbf{1})\right\}$

$l = 1, 2, 3, \ldots$

We can also present the spectrum in terms of complete $\mathcal{N}=1$ multiplets. The spectrum includes a single massless $U(1)$ vector multiplet. The massive spectrum is labelled by a positive integer $l = 1, 2, \ldots$ and includes the following states,

Mass	Degeneracy	Multiplet
$\frac{1}{R^2}\, l(l+1)$	$2l+1$	Massive Vector
$\frac{1}{R^2}\, l^2$	$4l$	Massive Chiral

4 Deconstruction

In this Section we motivate the appearance of extra dimensions in the large-N limit of $\mathcal{N}=1^*$ SUSY Yang-Mills with $U(N)$ gauge group. The $\mathcal{N}=1^*$ theory is a relevant deformation of the $\mathcal{N}=4$ SUSY Yang-Mills theory. We deform the $\mathcal{N}=4$ theory's superpotential by adding mass terms for the chiral multiplets,

$$\mathcal{W}(\Phi) = \mathrm{Tr}_N \left(i\sqrt{2}\,\Phi_1[\Phi_2, \Phi_3] + \eta \sum_{i=1}^{3} \Phi_i^2 \right) \tag{4.1}$$

The theory has no moduli space, instead it contains a number of isolated vacua [8]. The F-flatness condition is,

$$[\Phi_i, \Phi_j] = \sqrt{2}\eta\, i\varepsilon_{ijk}\Phi_k \tag{4.2}$$

Under the reparameterization,

$$\Phi_i \to \frac{1}{\sqrt{2}\eta}\Phi_i \tag{4.3}$$

the F-flatness condition (4.2) becomes,

$$[\Phi_i, \Phi_j] = i\varepsilon_{ijk}\Phi_k \tag{4.4}$$

which is precisely the $SU(2)$ Lie algebra. It can be solved by any N-dimensional representation of the $SU(2)$ generators, which in general will be reducible. This solution also satisfies the D-flatness condition. Our gauge group is $U(N) \sim SU(N) \times U(1)$, with Φ_i represented by $N \times N$ matrices. There is a single irreducible representation $J_i^{(d)}$ of the $SU(2)$ Lie algebra for every dimension d, which allows the gauge group to be decomposed into a number of irreducible representations, of total dimension N. If the number

of times a representation d appears is denoted k_d, then the unbroken gauge group is $U(N) \to \otimes_d U(k_d)$. We are interested in the Higgs vacuum where $\Phi_i = J_i^{(N)}$, breaking the gauge group $U(N) \to U(1)$. More general Higgs branches are present, where the gauge group is broken $U(N = pq) \to U(p)$ by p copies of the q-dimensional representation of the $SU(2)$ Lie algebra, $\Phi_i = \mathbb{1}_p \otimes J_i^{(q)}$.

The extra dimensions emerge via the mechanism seen in M(atrix) theory, which we will illustrate in the rest of this section. It was interpreted as deconstruction in [21]. We find that the Higgs vacuum describes a fuzzy sphere [22]. If we rescale the matrix fields[4] $\hat{x}_i = \tau\hat{\Phi}_i$, where $\tau^2 = \frac{4R^2}{N^2-1}$, then the expectation values in the Higgs vacuum satisfy,

$$\hat{x}_1^2 + \hat{x}_2^2 + \hat{x}_3^2 = \mathbb{1} \tag{4.5}$$

which is the defining equation of a fuzzy sphere. The coordinates of the fuzzy sphere $\hat{x}_i$ have the commutation relation,

$$[\hat{x}_i, \hat{x}_j] = i\tau\,\varepsilon_{ijk}\hat{x}_k \tag{4.6}$$

We recover the ordinary commutative sphere in the limit $N \to \infty$.

The Higgsed $\mathcal{N} = 1^*$ theory is a theory of $N \times N$ matrices. There is a well known correspondence between such matrix theories and non-commutative field theories. Scalar functions on a 2-sphere can be expanded in terms of spherical harmonics,

$$a(\theta, \phi) = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi) \tag{2.8}$$

The spherical harmonics can be expressed in terms of the cartesian coordinates x_A with $A = 1, 2, 3$ of a unit vector in $\Re^3$ [13],

$$Y_{lm}(\theta, \phi) = \sum_{\vec{A}} f^{(lm)}_{A_1 \ldots A_l} x^{A_1} \ldots x^{A_l} \tag{4.7}$$

where $f^{(lm)}_{A_1 \ldots A_l}$ is a traceless symmetric tensor of $SO(3)$ with rank l. Similarly we can expand $N \times N$ matrices of a matrix theory on a fuzzy sphere as,

$$\begin{aligned} \hat{a} &= \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm} && (4.8) \\ \hat{Y}_{lm} &= R^{-l} \sum_{\vec{A}} f^{(lm)}_{A_1 \ldots A_l} \hat{x}^{A_1} \ldots \hat{x}^{A_l} && (4.9) \end{aligned}$$

[4] We denote all matrix fields with hats.

where $\hat{x}_A = \frac{2R}{\sqrt{N^2-1}} J_A^{(N)}$ and $f^{(lm)}_{A_1 \ldots A_l}$ is the same tensor as in (4.7). The matrices $\hat{Y}_{lm}$ are known as fuzzy spherical harmonics. They obey the orthonormality condition,

$$\mathrm{Tr}_N \left(\hat{Y}^\dagger_{lm} \hat{Y}_{l'm'} \right) = \delta_{ll'} \, \delta_{mm'} \tag{4.10}$$

There is an obvious relation between equations (2.8) and (4.8).

$$\hat{a} = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} \hat{Y}_{lm} \to \ a(\theta, \phi) = \sum_{l=0}^{N-1} \sum_{m=-l}^{l} a_{lm} Y_{lm}(\theta, \phi) \tag{4.11}$$

Notice that the expansion in spherical harmonics is truncated at $N-1$ reflecting the finite number of degrees of freedom in the matrix $\hat{a}$. This is a 1:1 mapping, formally given by [13],

$$a(\theta, \phi) = \sum_{lm} \mathrm{Tr}_N (\hat{Y}^\dagger_{lm} \hat{a}) Y_{lm}(\theta, \phi) \tag{4.12}$$

The matrix trace is mapped by equation (4.11) to an integral over the sphere.

$$\frac{1}{N} \mathrm{Tr}_N \to \frac{1}{4\pi} \int d\Omega \tag{4.13}$$

The product of matrices maps to the star-product on the non-commutative sphere [5],

$$a * b(\theta, \phi) = \sum_{lm} \mathrm{Tr}_N (\hat{Y}^\dagger_{lm} \hat{a} \hat{b}) Y_{lm}(\theta, \phi) \tag{4.14}$$

This product is non-commutative due to the non-commutative nature of matrix multiplication. This mapping produces a correspondence between matrix theories and non-commutative field theories.

In analogy to continuum field theory we have derivative operators for the matrix theory. They correspond to the adjoint action of $J_i^{(N)}$ [13].

$$Ad(J_3^{(N)}) = \sum_{lm} a_{lm} \left[J_3^{(N)}, \hat{Y}_{lm} \right] = \sum_{lm} a_{lm} \, m \, \hat{Y}_{lm} \tag{4.15a}$$

$$\begin{aligned} Ad(J_\pm^{(N)}) &= \sum_{lm} a_{lm} \left[J_\pm^{(N)}, \hat{Y}_{lm} \right] \\ &= \sum_{lm} a_{lm} \sqrt{(l \pm m + 1)(l \mp m)} \, \hat{Y}_{lm\pm1} \end{aligned} \tag{4.15b}$$

[5] In order for the mapping to remain 1:1 we must assume that N is sufficiently large such that $l + l' \not> N - 1$.

The properties above, equations (4.15), show that by the correspondence between matrices and functions (4.11) the adjoint action of $J_i^{(N)}$ becomes,

$$Ad(J_i^{(N)}) \to L_i \tag{4.16}$$

The operator L_i is the derivative operator on the non-commutative sphere.

The fuzzy spherical harmonics have the commutator [23],

$$[Y_{l_1m_1}, Y_{l_2m_2}] = F_{l_1m_1l_2m_2}^{l_3m_3} Y_{l_3m_3} \tag{4.17}$$

where the structure constants are,

$$\begin{aligned} F_{l_1m_1l_2m_2}^{l_3m_3} = {} & 2\sqrt{(2l_1+1)(2l_2+1)(2l_3+1)}(-1)^{N-1} \\ & \times \begin{pmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \end{pmatrix} \begin{Bmatrix} l_1 & l_2 & l_3 \\ \frac{N-1}{2} & \frac{N-1}{2} & \frac{N-1}{2} \end{Bmatrix} \end{aligned} \tag{4.18}$$

$(\dots)$ is a Wigner 3j-symbol and $\{\dots\}$ is a Wigner 6j-symbol. For large-N the 6j-symbol behaves as $N^{-3/2}$ [23]. In the limit $N \to \infty$ the commutator (4.17) becomes,

$$[Y_{l_1m_1}, Y_{l_2m_2}] = 0 \tag{4.19}$$

and we recover the usual commutative spherical harmonics, this is the commutative limit.

The correspondence (4.11) allows us to map a matrix model to a non-commutative field theory. For the Higgsed $\mathcal{N} = 1^*$ theory this mechanism, from M(atrix) theory [13, 24], produces a six-dimensional non-commutative field theory, with a UV cutoff for finite N. The limit $N \to \infty$ the theory becomes a commutative, continuum field theory on $\Re^{3,1} \times S^2$.

5 Classical $\mathcal{N} = 1^*$ SUSY Yang-Mills Spectrum

In this section we will calculate the full classical spectrum of the Higgsed $\mathcal{N} = 1^*$ theory. The classical action of $\mathcal{N} = 1^*$ theory corresponding to the superpotential (4.1) with reparameterization (4.3) is,

$$\begin{aligned} \mathcal{S} = & \frac{1}{g_{ym}^2} \int d^4x \, \mathrm{Tr}_N \Big\{ -\frac{1}{4}\eta^2 F_{\mu\nu}F^{\mu\nu} - i\eta^3 \lambda\sigma^\mu D_\mu \bar{\lambda} - i\eta^3 \psi_i \sigma^\mu D_\mu \bar{\psi}_i \\ & - 2\eta^2 D_\mu \Phi_i^\dagger D^\mu \Phi_i + \eta^4 \Big(i\psi_i[\Phi_i^\dagger, \lambda] - i\lambda[\Phi_i^\dagger, \psi_i] + i\bar{\psi}_i[\Phi_i, \bar{\lambda}] - i\bar{\lambda}[\Phi_i, \bar{\psi}_i] \\ & + i\psi_i \, \varepsilon_{ijk}[\Phi_k, \psi_j] + i\bar{\psi}_i \, \varepsilon_{ijk}[\Phi_k^\dagger, \bar{\psi}_j] - \psi_i\psi_i - \bar{\psi}_i\bar{\psi}_i - 2[\Phi_i, \Phi_i^\dagger]^2 \\ & + 4[\Phi_i^\dagger, \Phi_j^\dagger][\Phi_i, \Phi_j] - 4i\varepsilon_{ijk}\Phi_i^\dagger[\Phi_j, \Phi_k] - 4i\varepsilon_{ijk}[\Phi_i^\dagger, \Phi_j^\dagger]\Phi_k - 8\,\Phi_i^\dagger\Phi_i \Big) \Big\} \end{aligned} \tag{5.1}$$

where λ is the gaugino and ψ_i are the superpartners of Φ_i; $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu + i\eta[A_\mu, A_\nu]$ and $D_\mu\Phi = \partial_\mu\Phi + i\eta[A_\mu, \Phi]$. We have chosen to normalise all our fields such that every term in the action has the prefactor $1/g_{ym}^2$. We expand the complex scalars about the Higgs vacuum $\hat{\Phi}_i = J_i^{(N)} + \delta\hat{\Phi}_i$, only terms quadratic in the fields contribute to the mass spectrum, hence we will ignore the higher orders. We will derive the spectrum from both the fermionic and bosonic contributions. We first calculate the fermionic mass matrix, the contribution is,

$$\begin{aligned}\mathcal{L}_{FM} = 2\eta\,\mathrm{Tr}_N\Big\{ & i\hat{\psi}_i\,\varepsilon_{ijk}[J_k, \hat{\psi}_j] + i\hat{\bar{\psi}}_i\varepsilon_{ijk}\,[J_k, \hat{\bar{\psi}}_j] - i\hat{\lambda}[J_i, \hat{\psi}_i] \\ & + i\hat{\psi}_i[J_i, \hat{\lambda}] + i\hat{\bar{\psi}}_i[J_i, \hat{\bar{\lambda}}] - i\hat{\bar{\lambda}}[J_i, \hat{\bar{\psi}}_i] - \hat{\psi}_i\hat{\psi}_i - \hat{\bar{\psi}}_i\hat{\bar{\psi}}_i\Big\}\end{aligned} \tag{5.2}$$

where we have suppressed the label (N). We rewrite this expression to form the fermionic mass matrices,

$$\mathcal{L}_{FM} = 2\eta\Big\{\big(\hat{\Psi}_R\big)_{ab}\Delta^{(RS)}_{ab,cd}\big(\hat{\Psi}^T_S\big)_{cd} + \big(\hat{\bar{\Psi}}_R\big)_{ab}\bar{\Delta}^{(RS)}_{ab,cd}\big(\hat{\bar{\Psi}}^T_S\big)_{cd}\Big\} \tag{5.3}$$

where the four species of Weyl fermion are combined in a column $\hat{\Psi}_R$ with $\hat{\Psi}_R = \hat{\psi}_i$ for $R = i = 1, 2, 3$ and $\hat{\Psi}_4 = \hat{\lambda}$. The mass matrices Δ and $\bar{\Delta}$ have the explicit form,

$$\Delta^{(ij)}_{ab,cd} = i\varepsilon_{ijk}\Big(\delta_{ac}(J_k)_{bd} - (J_k^*)_{ac}\,\delta_{bd}\Big) - \delta_{ij}\,\delta_{ac}\,\delta_{bd} = \bar{\Delta}^{(ij)}_{ab,cd} \tag{5.4a}$$

$$\Delta^{(i4)}_{ab,cd} = -i\Big((J_i^*)_{ac}\,\delta_{bd} - \delta_{ac}(J_i)_{bd}\Big) = \bar{\Delta}^{(i4)}_{ab,cd} \tag{5.4b}$$

$$\Delta^{(4i)}_{ab,cd} = -i\Big(\delta_{ac}(J_i)_{bd} - (J_i^*)_{ac}\,\delta_{bd}\Big) = \bar{\Delta}^{(4i)}_{ab,cd} \tag{5.4c}$$

The masses of physical states are detemined by the squared mass matrix,

$$M^{(RS)}_{ab,\,ef} = 4\eta^2\bar{\Delta}^{(RT)}_{ab,\,cd}\,\Delta^{(TS)}_{cd,\,ef} \tag{5.5}$$

Explicitly these matrices are,

$$\begin{aligned}
M^{(ij)}_{ab,ef} &= (J^*_i J^*_j - J^*_j J^*_i)_{ae}\delta_{bf} + \delta_{ae}(J_i J_j - J_j J_i)_{bf} \\
&\quad -2i\varepsilon_{ijk}\Big(\delta_{ae}(J_k)_{bf} - (J^*_k)_{ae}\delta_{bf}\Big) + \delta_{ij}\Big\{\delta_{ae}(J_k J_k)_{bf} \qquad (5.6a)\\
&\quad -2(J^*_k)_{ae}(J_k)_{bf} + (J^*_k J^*_k)_{ae}\delta_{bf} + \delta_{ae}\delta_{bf}\Big\} \\
M^{(i4)}_{ab,ef} &= \varepsilon_{ijk}\Big((J^*_j)_{ae}(J_k)_{bf} - (J^*_k J^*_j)_{ae}\delta_{bf} - \delta_{ae}(J_k J_j)_{bf} \qquad (5.6b)\\
&\quad +(J^*_k)_{ae}(J_j)_{bf}\Big) + i\Big((J^*_i)_{ae}\delta_{bf} - \delta_{ae}(J_i)_{bf}\Big) \\
M^{(4i)}_{ab,ef} &= -\varepsilon_{ijk}\Big\{\delta_{ae}(J_j J_k)_{bf} - (J^*_k)_{ae}(J_j)_{bf} - (J^*_j)_{ae}(J_k)_{bf} \qquad (5.6c)\\
&\quad +(J^*_j J^*_k)_{ae}\delta_{bf}\Big\} - i\Big((J^*_i)_{ae}\delta_{bf} - \delta_{ae}(J_i)_{bf}\Big) \\
M^{(44)}_{ab,ef} &= \delta_{ae}(J_i J_i)_{bf} - 2(J^*_i)_{ae}(J_i)_{bf} + (J^*_i J^*_i)_{ae}\delta_{bf} \qquad (5.6d)
\end{aligned}$$

In order to diagonalize this matrix we consider the bilinear form,

$$\mathcal{M}_F = \big(\hat{\Psi}^\dagger_R\big)_{ab} M^{(RS)}_{ab,\,ef}\big(\hat{\Psi}^T_S\big)_{ef} \tag{5.7}$$

We expand the fermionic fields $\hat{\Psi}_R$ in fuzzy spherical harmonics as,

$$\hat{\Psi}_R = \sum_{lm} \Psi^{(R)}_{lm}\hat{Y}_{lm} \tag{5.8}$$

This expansion then yields,

$$\mathcal{M}_F = 4\eta^2 \sum_{l=0}^{N-1}\sum_{m=-l}^{l}\sum_{l'=0}^{N-1}\sum_{m'=-l'}^{l'} \big(\Psi^{(R)}_{lm}\big)^\dagger \Psi^{(S)}_{l'm'} N^{(RS)}_{lm,\,l'm'} \tag{5.9}$$

with

$$N^{(RS)}_{lm,\,l'm'} = \delta_{ll'} \begin{pmatrix} J^2_{(L)}+1 & -iJ^{(L)}_3 & iJ^{(L)}_2 & 0 \\ iJ^{(L)}_3 & J^2_{(L)}+1 & -iJ^{(L)}_1 & 0 \\ -iJ^{(L)}_2 & iJ^{(L)}_1 & J^2_{(L)}+1 & 0 \\ 0 & 0 & 0 & J^2_{(L)} \end{pmatrix}_{mm'} \tag{5.10}$$

with $L = 2l+1$.

The bosonic mass matrix receives a contribution from the scalar potential and a contribution from the covariant derivative of the complex scalars. The scalar potential of the $\mathcal{N} = 1^*$ theory is,

$$V = 4\eta^4 \mathrm{Tr}_N \hat{H}^\dagger_{ij}\hat{H}_{ij} + 2\eta^4 \mathrm{Tr}_N \hat{D}^2 \tag{5.11}$$

where,

$$\begin{aligned}\hat{H}_{ij} &= [\hat{\Phi}_i, \hat{\Phi}_j] - i\varepsilon_{ijk}\,\hat{\Phi}_k \\ &= [J_i, \delta\hat{\Phi}_j] - [J_j, \delta\hat{\Phi}_i] + [\delta\hat{\Phi}_i, \delta\hat{\Phi}_j] - i\varepsilon_{ijk}\,\delta\hat{\Phi}_k\end{aligned}$$

$$\begin{aligned}\hat{D}^2 &= [\hat{\Phi}_i^\dagger, \hat{\Phi}_i]^2 \\ &= \Big([J_i, \delta\hat{\Phi}_i^\dagger] + [J_i, \delta\hat{\Phi}_i] + [\delta\hat{\Phi}_i^\dagger, \delta\hat{\Phi}_i]\Big)^2\end{aligned}$$

The scalar potential to quadratic order in the fields is,

$$\begin{aligned}V = 2\eta^4 \mathrm{Tr}_N \Big\{ &-4[J_i, \delta\hat{\Phi}_j^\dagger][J_i, \delta\hat{\Phi}_j] + 4[J_i, \delta\hat{\Phi}_j^\dagger][J_j, \delta\hat{\Phi}_i] + 8i\varepsilon_{ijk}\delta\hat{\Phi}_k^\dagger[J_i, \delta\hat{\Phi}_j] \\ &+ 4\delta\hat{\Phi}_i^\dagger\delta\hat{\Phi}_i + [J_i, \delta\hat{\Phi}_i^\dagger]^2 + 2[J_i, \delta\hat{\Phi}_i^\dagger][J_j, \delta\hat{\Phi}_j] + [J_i, \delta\hat{\Phi}_i]^2\Big\}\end{aligned} \tag{5.12}$$

Using the cyclicity of the trace and imposing the gauge condition $[J_i, \delta\hat{\Phi}_i] = 0$ the second term becomes,

$$\begin{aligned}\mathrm{Tr}_N[J_i, \delta\hat{\Phi}_j^\dagger][J_j, \delta\hat{\Phi}_i] &= \mathrm{Tr}_N\Big([\delta\hat{\Phi}_j^\dagger, J_j][\delta\hat{\Phi}_i, J_i] - [\delta\hat{\Phi}_j^\dagger, \delta\hat{\Phi}_i][J_j, J_i]\Big) \\ &= -\mathrm{Tr}_N\, i\varepsilon_{ijk}\,\delta\hat{\Phi}_k^\dagger[J_i, \delta\hat{\Phi}_j]\end{aligned} \tag{5.13}$$

Hence under the gauge condition the scalar potential is reduced to,

$$V = 8\eta^4 \mathrm{Tr}_N\Big\{\delta\hat{\Phi}_j^\dagger[J_i, [J_i, \delta\hat{\Phi}_j]] + i\varepsilon_{ijk}\,\delta\hat{\Phi}_i^\dagger[J_j, \delta\hat{\Phi}_k] + \delta\hat{\Phi}_i^\dagger\delta\hat{\Phi}_i\Big\} \tag{5.14}$$

The bosonic mass contribution is,

$$\mathcal{M}_V = 4\eta^2 \mathrm{Tr}_N\Big\{\delta\hat{\Phi}_j^\dagger[J_i, [J_i, \delta\hat{\Phi}_j]] + i\varepsilon_{ijk}\,\delta\hat{\Phi}_i^\dagger[J_j, \delta\hat{\Phi}_k] + \delta\hat{\Phi}_i^\dagger\delta\hat{\Phi}_i\Big\} \tag{5.15}$$

Expanding perturbations $\delta\hat{\Phi}_i$ in fuzzy spherical harmonics,

$$\delta\hat{\Phi}_i = \sum_{l,m} \phi_{lm}^{(i)}\hat{Y}_{lm} \tag{5.16a}$$

$$\delta\hat{\Phi}_i^\dagger = \sum_{l,m} \bar{\phi}_{lm}^{(i)}\hat{Y}_{lm}^\dagger \tag{5.16b}$$

then,

$$\mathcal{M}_V = 4\eta^2 \sum_{l,m,l',m'} \left(\phi_{lm}^i\right)^\dagger \phi_{l'm'}^k N_{lm,l'm'}^{(ik)} \tag{5.17}$$

where,

$$N^{(ik)}_{lm,l'm'} = \left(J^2_{(L)} + 1\right)_{mm'} \delta_{ll'}\delta_{ik} + i\varepsilon_{ijk}\left(J^{(L)}_j\right)_{mm'} \delta_{ll'} \tag{5.18}$$

The covariant derivative of the complex scalar is,

$$2\eta^2 \mathrm{Tr}_N D_\mu \hat{\Phi}^\dagger_i D^\mu \hat{\Phi}_i = 2\eta^2 \mathrm{Tr}_N \left(\partial_\mu \hat{\Phi}^\dagger_i + i\eta[A_\mu, \hat{\Phi}^\dagger_i]\right)\left(\partial^\mu \hat{\Phi}_i + i\eta[A^\mu, \hat{\Phi}_i]\right) \tag{5.19}$$

This contributes to the bosonic mass matrix,

$$\begin{aligned}\mathcal{M}_D = -4\eta^2\, \mathrm{Tr}_N[J_i, \hat{A}_\mu][J_i, \hat{A}^\mu] &= 4\eta^2\, \mathrm{Tr}_N \hat{A}_\mu [J^2, \hat{A}^\mu] \\ &= 4\eta^2 \sum_{lm,l'm'} a_{(\mu)\,lm} a^{(\mu)}_{l'm'} \left(J^2_{(L)}\right)_{mm'} \delta_{ll'}\end{aligned} \tag{5.20}$$

The Bosonic Mass Matrix is,

$$\mathcal{M}_B = \left(\hat{\Phi}^\dagger_R\right)_{ab} M^{(RS)}_{ab,\,ef} \left(\hat{\Phi}^T_S\right)_{ef} = 4\eta^2 \sum_{l=0}^{N-1} \sum_{m=-l}^{l} \sum_{l'=0}^{N-1} \sum_{m'=-l'}^{l'} \left(\phi^R_{lm}\right)^\dagger \phi^S_{l'm'} N^{(RS)}_{lm,\,l'm'} \tag{5.21}$$

where the three complex scalars and gauge boson are combined in a column vector $\hat{\Phi}_R$ with $\hat{\Phi}_R = \hat{\Phi}_i$ for $i = 1, 2, 3$ and $\hat{\Phi}_4 = \hat{A}_\mu$. The matrix $N^{(RS)}_{lm,\,l'm'}$ is the same matrix as (5.10).

The fermionic and bosonic calculations lead to the same matrix. To complete the diagonalization we must solve the characteristic equation of the matrix $N^{(RS)}_{lm,\,l'm'}$. Consider the $(p+q) \times (p+q)$ matrix,

$$\mathcal{X} = \begin{array}{cc} & \begin{array}{cc} (p) & (q) \end{array} \\ \begin{array}{c} (p) \\ (q) \end{array} & \begin{pmatrix} A & B \\ C & D \end{pmatrix} \end{array} \tag{5.22}$$

The determinant of $\mathcal{X}$ can be evaluated using the formula,

$$Det(\mathcal{X}) = Det(A) Det(D - CA^{-1}B) \tag{5.23}$$

Clearly in the mass matrix $N^{(RS)}_{lm,\,l'm'}$ the gauge boson/gaugino contribution is trivial and we will concentrate on calculating the determinant of,

$$\tilde{N}_{mm'} = \begin{pmatrix} \gamma^{(L)}.\mathbb{1} & -iJ^{(L)}_3 & iJ^{(L)}_2 \\ iJ^{(L)}_3 & \gamma^{(L)}.\mathbb{1} & -iJ^{(L)}_1 \\ -iJ^{(L)}_2 & iJ^{(L)}_1 & \gamma^{(L)}.\mathbb{1} \end{pmatrix}_{mm'} \tag{5.24}$$

where $\gamma^{(L)} = l(l+1) + 1 - \lambda$ and λ is the eigenvalue of the characteristic equation,

$$\mathrm{Det}\left(N - \lambda \mathbb{1}\right) = 0 \tag{5.25}$$

We define the matrix,

$$A = \begin{pmatrix} \gamma^{(L)}.\mathbb{1} & -iJ_3^{(L)} \\ iJ_3^{(L)} & \gamma^{(L)}.\mathbb{1} \end{pmatrix} \tag{5.26}$$

whose inverse is,

$$A^{-1} = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \tag{5.27}$$

where,

$$a = \frac{\gamma^{(L)}}{(\gamma^{(l)})^2 - m^2}\,\delta_{mm'} \tag{5.28a}$$

$$b = \frac{im}{(\gamma^{(L)})^2 - m^2}\,\delta_{mm'} \tag{5.28b}$$

We find that,

$$Det(N-\lambda\mathbb{1}) = \prod_{l=0}^{N-1}\prod_{m=-l}^{l}(\gamma^{(L)}-1)^{2(2l+1)}(\gamma^{(L)}+l)^{2l+3}\big(\gamma^{(L)}-(l+1)\big)^{2l-1} \tag{5.29}$$

The roots of this characteristic equation yield the eigenvalues of the mass matrices.

For $l = 0$ we therefore find,

Eigenvalue	Degeneracy
0	1
1	3

while for $l = 1, 2, \ldots, N-1$ we get,

Eigenvalue	Degeneracy
l^2	$2l-1$
$l(l+1)$	$2(2l+1)$
$(l+1)^2$	$2l+3$

To find the complete spectrum in this case we sum over all values of l. The final result is a single zero eigenvalue and two series of eigenvalues labeled by a positive integer $k = 1, 2, \ldots N-1$,

Eigenvalue	Degeneracy
k^2	$4k$
$k(k+1)$	$2(2k+1)$

Finally we find one extra eigenvalue $\lambda = N^2$ with degeneracy $2N+1$.

Each eigenvalue of the fermionic mass matrix corresponds to a single left-handed Weyl fermion and it's right-handed charge conjugate. Similarly each eigenvalue of the bosonic matrix corresponds to a complex scalar or gauge boson. The theory has $\mathcal{N}=1$ supersymmetry, therefore all the states must form $\mathcal{N}=1$ supersymmetry multiplets. We began with a $U(N)$ gauge group which is broken to $U(1)$. The spectrum must contain a single massless gauge boson and N^2-1 massive gauge bosons. Clearly the $l=0$ state must form a massless vector multiplet. The N^2-1 massive gauge bosons must form N^2-1 massive vector multiplets, which are formed from a massless vector multiplet and a massless chiral multiplet. The remaining states must form massive chiral multiplets.

In addition to a single massless vector multiplet we have two towers of multiplets labelled by $k=1,2,\ldots,N-1$ as tabulated below,

M^2	Degeneracy	Multiplet
$\eta^2 k(k+1)$	$2k+1$	Massive Vector
$\eta^2 k^2$	$4k$	Massive Chiral

The spectrum is completed by $2N+1$ chiral multiplets of mass $M^2=\eta^2 N^2$. In the limit $N\to\infty$ this precisely matches the spectrum of the Maldacena-Núñez compactification with the identification $\eta \sim \frac{1}{R}$, up to a numerical coefficient. For finite N the spectrum of the $\mathcal{N}=1^*$ theory is a subset of that of the six-dimensional theory obtained by retaining only those states with mass less than N^2 (together with $2N+1$ chiral multiplets of mass equal to N^2).

We can generalise the above calculation to the more general Higgs vacua discussed in section 4, where the gauge group is broken from $U(N)\to U(p)$ with $N=pq$. In this case, the vacuum expectation value takes the form,

$$< \Phi_i >= \mathbb{1}_{(p)} \otimes J_i^{(q)} \tag{5.30}$$

In this vacuum the field fluctuations decompose as a tensor product,

$$\delta\hat{\Phi}_i = \sum_{l,m} \phi^{(i)}_{lm(p)} \otimes \hat{Y}^{(q)}_{lm} \tag{5.31}$$

The q^2 coefficients $\phi^{(i)}_{lm}$ are $p\times p$ matrices and the fuzzy spherical harmonics are $q\times q$ matrices. The $k=0$ mode remains a massless vector multiplet, but now has degeneracy p^2. Note that, in constrast to the maximally Higgsed vacuum, the $U(p)$ low-energy effective theory is asymptotically free and a classical analysis is only reliable at scales much higher than the corresponding dynamical scale.

The remaining q^2-1 modes (at finite N, q) are,

M^2	Degeneracy	Multiplet
$\eta^2 k(k+1)$	$(2k+1)p^2$	Massive Vector
$\eta^2 k^2$	$4k\,p^2$	Massive Chiral

for $k=1,2,\ldots,q-1$. The spectrum is completed by $(2q+1)p^2$ extra massive chiral multiplets with mass $M^2=\eta^2 q^2$. Notice that the degeneracies of all states are integer multiples of p^2 consistent with each state transforming in the adjoint of the unbroken $U(p)$ gauge symmetry. Finally, we can also study the spectrum corresponding to vacua of the $SU(N)$ theory. The effect is simply to replace p^2 by p^2-1 appropriate for adjoint multiplets of an unbroken $SU(p)$.

In each of the cases discussed above, the spectrum matches the Kaluza-Klein spectrum of a six-dimensional theory with non-abelian gauge group ($U(p)$ for the $U(N)$ theory and $SU(p)$ for $SU(N)$). In fact this spectrum can be obtained from a MN twisted compactification of the corresponding $\mathcal{N}=(1,1)$ SUSY Yang-Mills theory by an easy generalisation of the analysis of Section 3.

6 Effective Six-Dimensional Theory

In the $N\to\infty$ limit, the classical spectrum of the $\mathcal{N}=1^*$ theory is identical to that of the MN compactification. We now calculate the effective action of the Higgsed $\mathcal{N}=1^*$ theory and compare the action with the six-dimensional bosonic action (3.13) action derived in section 3.1. We follow the deconstruction procedure described in section 4, mapping the four-dimensional matrix model of the Higgsed $\mathcal{N}=1^*$ theory to a six-dimensional non-commutative field theory, and then take the commutative limit ($N\to\infty$). We will break up the calculation into four parts. We begin with the scalar potential, with gauge condition $[J_i,\delta\hat{\Phi}_i]=0$ imposed.

$$\begin{aligned}
V &= 4\eta^4\mathrm{Tr}_N\hat{H}^\dagger_{ij}\hat{H}_{ij}+2\eta^4\mathrm{Tr}_N\hat{D}^2\\
&= 8\eta^4\mathrm{Tr}_N\Big\{\delta\hat{\Phi}^\dagger_j[J_i,[J_i,\delta\hat{\Phi}_j]]+i\varepsilon_{ijk}\delta\hat{\Phi}^\dagger_i[J_j,\delta\hat{\Phi}_k]+\delta\hat{\Phi}^\dagger_i\delta\hat{\Phi}_i\\
&\quad -[J_i,\delta\hat{\Phi}_j][\delta\hat{\Phi}^\dagger_j,\delta\hat{\Phi}_i]+[J_i,\delta\hat{\Phi}^\dagger_j][\delta\hat{\Phi}^\dagger_i,\delta\hat{\Phi}_j]+i\varepsilon_{ijk}[\delta\hat{\Phi}^\dagger_i,\delta\hat{\Phi}^\dagger_j]\delta\hat{\Phi}_k\\
&\quad +i\varepsilon_{ijk}\delta\hat{\Phi}^\dagger_i[\delta\hat{\Phi}_j,\delta\hat{\Phi}_k]-[\delta\hat{\Phi}^\dagger_i,\delta\hat{\Phi}^\dagger_j][\delta\hat{\Phi}_i,\delta\hat{\Phi}_j]+\frac{1}{4}[\delta\hat{\Phi}^\dagger_i,\delta\hat{\Phi}_i]^2\Big\}
\end{aligned}$$

where,

$$\begin{aligned}
\hat{H}_{ij} &= [\hat{\Phi}_i, \hat{\Phi}_j] - i\varepsilon_{ijk}\,\hat{\Phi}_k \\
&= [J_i, \delta\hat{\Phi}_j] - [J_j, \delta\hat{\Phi}_i] + [\delta\hat{\Phi}_i, \delta\hat{\Phi}_j] - i\varepsilon_{ijk}\,\delta\hat{\Phi}_k \\
\hat{D} &= [J_i, \delta\hat{\Phi}_i^\dagger] + [J_i, \delta\hat{\Phi}_i] + [\delta\hat{\Phi}_i^\dagger, \delta\hat{\Phi}_i]
\end{aligned}$$

From the correspondence between matrices and functions (equations (4.11) and (4.13)),

$$\begin{aligned}
V &= 8\frac{N}{4\pi}\,\eta^4 \int d\Omega \Big\{ \delta\Phi_i^\dagger \left(L^2\,\delta\Phi_i \right) + i\varepsilon_{ijk}\delta\Phi_i^\dagger \left(L_j\,\delta\Phi_k \right) + \delta\Phi_i^\dagger\delta\Phi_i \\
&\quad -(L_i\delta\Phi_j)[\delta\Phi_j^\dagger, \delta\Phi_i] + (L_i\delta\Phi_j^\dagger)[\delta\Phi_i^\dagger, \delta\Phi_j] + i\varepsilon_{ijk}[\delta\Phi_i^\dagger, \delta\Phi_j^\dagger]\delta\Phi_k \\
&\quad + i\varepsilon_{ijk}\delta\Phi_i^\dagger[\delta\Phi_j, \delta\Phi_k] - [\delta\Phi_i^\dagger, \delta\Phi_j^\dagger][\delta\Phi_i, \delta\Phi_j] + \frac{1}{4}[\delta\Phi_i^\dagger, \delta\Phi_i]^2 \Big\}_\star
\end{aligned}$$

where $\{\}_\star$ means all products are non-commutative star-products. Taking the commutative limit [6],

$$V = 8\frac{N}{4\pi}\,\eta^4 \int d\Omega \Big\{ \delta\Phi_i^\dagger \left(L^2\,\delta\Phi_i \right) + i\varepsilon_{ijk}\delta\Phi_i^\dagger \left(L_j\,\delta\Phi_k \right) + \delta\Phi_i^\dagger\delta\Phi_i \Big\} \tag{6.1}$$

The bosonic kinetic terms are,

$$\begin{aligned}
\mathcal{L}_{Bkin} &= \eta^2 \mathrm{Tr}_N \left\{ -\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - 2D_\mu\hat{\Phi}_i^\dagger D^\mu\hat{\Phi}_i \right\} \\
&= \eta^2 \mathrm{Tr}_N \Big\{ -\frac{1}{4}\hat{F}_{\mu\nu}\hat{F}^{\mu\nu} - 2\Big(\partial_\mu\delta\hat{\Phi}_i^\dagger\partial^\mu\delta\hat{\Phi}_i - i\eta[J_i, \hat{A}_\mu]\partial^\mu\delta\hat{\Phi}_i \\
&\quad + i\eta[\hat{A}_\mu, \delta\hat{\Phi}_i^\dagger]\partial^\mu\delta\hat{\Phi}_i - i\eta\partial_\mu\delta\hat{\Phi}_i^\dagger[J_i, \hat{A}^\mu] + i\eta\partial_\mu\delta\hat{\Phi}_i^\dagger[\hat{A}^\mu, \delta\hat{\Phi}_i] \\
&\quad - \eta^2[J_i, \hat{A}_\mu][J_i, \hat{A}^\mu] + \eta^2[J_i, \hat{A}_\mu][\hat{A}^\mu, \delta\hat{\Phi}_i] + \eta^2[\hat{A}_\mu, \delta\hat{\Phi}_i^\dagger][J_i, \hat{A}^\mu] \\
&\quad - \eta^2[\hat{A}_\mu, \delta\hat{\Phi}_i^\dagger][\hat{A}^\mu, \delta\hat{\Phi}_i] \Big) \Big\}
\end{aligned}$$

where $\hat{F}_{\mu\nu} = \partial_\mu\hat{A}_\nu - \partial_\nu\hat{A}_\mu + i\eta[\hat{A}_\mu, \hat{A}_\nu]$ and $D_\mu\hat{\Phi}_i = \partial_\mu\hat{\Phi}_i + i\eta[\hat{A}_\mu, \hat{\Phi}_i]$. From the correspondence between matrices and functions,

$$\begin{aligned}
\mathcal{L}_{Bkin} &= \frac{N}{4\pi}\,\eta^2 \int d\Omega \Big\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\Big(\partial_\mu(\delta\Phi_i^\dagger)\partial^\mu(\delta\Phi_i) - i\eta\partial_\mu(\delta\Phi_i^\dagger)(L_iA^\mu) \\
&\quad + i\eta\partial_\mu(\delta\Phi_i^\dagger)[A^\mu, \delta\Phi_i] - i\eta(L_iA_\mu)\partial^\mu(\delta\Phi_i) + i\eta[A_\mu, \delta\Phi_i^\dagger]\partial^\mu(\delta\Phi_i) \\
&\quad - \eta^2(L_iA_\mu)(L_iA^\mu) + \eta^2(L_iA_\mu)[A^\mu, \delta\Phi_i] + \eta^2[A_\mu, \delta\Phi_i^\dagger](L_iA^\mu) \\
&\quad + \eta^2[A_\mu, \delta\Phi_i^\dagger][A^\mu, \delta\Phi_i] \Big) \Big\}_\star
\end{aligned}$$

[6] Note commutators of field fluctuations vanish in this limit, see equation (4.19), e.g. $[\delta\Phi_i, \delta\Phi_j] = 0$

Taking the commutative limit and applying the gauge condition $L_i\delta\Phi_i = 0$,

$$\mathcal{L}_{Bkin} = \frac{N}{4\pi}\,\eta^2 \int d\Omega\Big\{-\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\partial_\mu(\delta\Phi_i^\dagger)\partial^\mu(\delta\Phi_i) + 2\eta^2(L_iA_\mu)(L_iA^\mu)\Big\} \tag{6.2}$$

where now $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$. Consider the fermionic interaction terms,

$$\begin{aligned}
\mathcal{L}_{Fint} &= \eta^4 \mathrm{Tr}_N\Big\{i\hat{\psi}_i\epsilon_{ijk}[\hat{\Phi}_k,\hat{\psi}_j] + i\hat{\bar{\psi}}_i\epsilon_{ijk}[\hat{\Phi}_k^\dagger,\hat{\bar{\psi}}_j] - i\hat{\lambda}[\hat{\Phi}_i^\dagger,\hat{\psi}_i] \\
&\quad + i\hat{\psi}_i[\hat{\Phi}_i^\dagger,\hat{\lambda}] + i\hat{\bar{\psi}}_i[\hat{\Phi}_i,\hat{\bar{\lambda}}] - i\hat{\bar{\lambda}}[\hat{\Phi}_i,\hat{\bar{\psi}}_i] - \hat{\psi}_i\hat{\psi}_i - \hat{\bar{\psi}}_i\hat{\bar{\psi}}_i\Big\} \\
&= \eta^4 \mathrm{Tr}_N\Big\{i\hat{\psi}_i\epsilon_{ijk}[J_k,\hat{\psi}_j] + i\hat{\psi}_i\epsilon_{ijk}[\delta\hat{\Phi}_k,\hat{\psi}_j] + i\hat{\bar{\psi}}_i\epsilon_{ijk}[J_k,\hat{\bar{\psi}}_j] \\
&\quad + i\hat{\bar{\psi}}_i\epsilon_{ijk}[\delta\hat{\Phi}_k^\dagger,\hat{\bar{\psi}}_j] - i\hat{\lambda}[J_i,\hat{\psi}_i] - i\hat{\lambda}[\delta\hat{\Phi}_i^\dagger,\hat{\psi}_i] + i\hat{\psi}_i[J_i,\hat{\lambda}] + i\hat{\psi}_i[\delta\hat{\Phi}_i^\dagger,\hat{\lambda}] \\
&\quad + i\hat{\bar{\psi}}_i[J_i,\hat{\bar{\lambda}}] + i\hat{\bar{\psi}}_i[\delta\hat{\Phi}_i,\hat{\bar{\lambda}}] - i\hat{\bar{\lambda}}[J_i,\hat{\bar{\psi}}_i] - i\hat{\bar{\lambda}}[\delta\hat{\Phi}_i,\hat{\bar{\psi}}_i] - \hat{\psi}_i\hat{\psi}_i - \hat{\bar{\psi}}_i\hat{\bar{\psi}}_i\Big\}
\end{aligned}$$

From the correspondence between matrices and functions,

$$\begin{aligned}
\mathcal{L}_{Fint} &= \frac{N}{4\pi}\,\eta^4 \int d\Omega\Big\{i\psi_i\epsilon_{ijk}\left(L_k\psi_j\right) + i\psi_i\epsilon_{ijk}[\delta\Phi_k,\psi_j] + i\bar{\psi}_i\epsilon_{ijk}\left(L_k\bar{\psi}_j\right) \\
&\quad + i\bar{\psi}_i\epsilon_{ijk}[\delta\Phi_k^\dagger,\bar{\psi}_j] - i\lambda\left(L_i\psi_i\right) - i\lambda[\delta\Phi_i^\dagger,\psi_i] + i\psi_i\left(L_i\lambda\right) + i\psi_i[\delta\Phi_i^\dagger,\lambda] \\
&\quad + i\bar{\psi}_i\left(L_i\bar{\lambda}\right) + i\bar{\psi}_i[\delta\Phi_i,\bar{\lambda}] - i\bar{\lambda}\left(L_i\bar{\psi}_i\right) - i\bar{\lambda}[\delta\Phi_i,\bar{\psi}_i] - \psi_i\psi_i - \bar{\psi}_i\bar{\psi}_i\Big\}_\star
\end{aligned}$$

In the commutative limit,

$$\begin{aligned}
\mathcal{L}_{Fint} = \frac{N}{4\pi}\,\eta^4 \int d\Omega\Big\{&i\psi_i\,\varepsilon_{ijk}L_k\psi_j + i\bar{\psi}_i\,\varepsilon_{ijk}L_k\bar{\psi}_j - i\lambda L_i\psi_i \\
&+ i\psi_iL_i\lambda + i\bar{\psi}_iL_i\bar{\lambda} - i\bar{\lambda}L_i\bar{\psi}_i - \psi_i\psi_i - \bar{\psi}_i\bar{\psi}_i\Big\}
\end{aligned} \tag{6.3}$$

The fermionic kinetic terms are,

$$\begin{aligned}
\mathcal{L}_{Fkin} &= -i\eta^3\mathrm{Tr}_N\Big(\hat{\lambda}\sigma^\mu D_\mu\hat{\bar{\lambda}} + \hat{\psi}_i\sigma^\mu D_\mu\hat{\bar{\psi}}_i\Big) \\
&= -i\eta^3\mathrm{Tr}_N\Big(\hat{\lambda}\sigma^\mu\partial_\mu\hat{\bar{\lambda}} + i\eta\hat{\lambda}\sigma^\mu[\hat{A}_\mu,\hat{\bar{\lambda}}] + \hat{\psi}_i\sigma^\mu\partial_\mu\hat{\bar{\psi}}_i + i\eta\hat{\psi}_i\sigma^\mu[\hat{A}_\mu,\hat{\bar{\psi}}_i]\Big)
\end{aligned}$$

From the correspondence between matrices and functions,

$$\begin{aligned}
\mathcal{L}_{Fkin} = -i\frac{N}{4\pi}\,\eta^3 \int d\Omega\Big(&\lambda\sigma^\mu\partial_\mu\bar{\lambda} + i\eta\lambda\sigma^\mu[A_\mu,\bar{\lambda}] + \psi_i\sigma^\mu\partial_\mu\bar{\psi}_i \\
&+ i\eta\psi_i\sigma^\mu[A_\mu,\bar{\psi}_i]\Big)_\star
\end{aligned}$$

In the commutative limit,

$$\mathcal{L}_{Fkin} = -i\frac{N}{4\pi}\,\eta^3 \int d\Omega\Big(\lambda\sigma^\mu\partial_\mu\bar{\lambda} + \psi_i\sigma^\mu\partial_\mu\bar{\psi}_i\Big) \tag{6.4}$$

Summarising the action,

$$\begin{aligned}\mathcal{S} =& \frac{1}{g_{ym}^2}\frac{N}{4\pi}\,\eta^2 \int d^4x \int d\Omega\Big\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\partial_\mu(\delta\Phi_i^\dagger)\partial^\mu(\delta\Phi_i) \\ & - i\eta\lambda\sigma^\mu\partial_\mu\bar{\lambda} - i\eta\psi_i\sigma^\mu\partial_\mu\bar{\psi}_i + \eta^2\Big(2(L_iA_\mu)(L_iA^\mu) + i\psi_i\varepsilon_{ijk}\,L_k\psi_j \\ & + i\bar{\psi}_i\varepsilon_{ijk}\,L_k\bar{\psi}_j - i\lambda L_i\psi_i + i\psi_iL_i\lambda + i\bar{\psi}_iL_i\bar{\lambda} - i\bar{\lambda}L_i\bar{\psi}_i - \psi_i\psi_i \\ & - \bar{\psi}_i\bar{\psi}_i - 8\delta\Phi_i^\dagger\Big(L^2\,\delta\Phi_i\Big) - 8i\varepsilon_{ijk}\,\delta\Phi_i^\dagger\Big(L_j\,\delta\Phi_k\Big) - 8\delta\Phi_i^\dagger\delta\Phi_i\Big)\Big\}\end{aligned} \tag{6.5}$$

Before proceeding further we want to rewrite this action with Majorana spinors. We define the Majorana spinors,

$$\Psi_{iA} = \begin{pmatrix} \psi_{i\alpha} \\ \bar{\psi}_i^{\,\dot{\alpha}} \end{pmatrix} \qquad\qquad \Lambda_A = \begin{pmatrix} \lambda_\alpha \\ \bar{\lambda}_i^{\dot{\alpha}} \end{pmatrix} \tag{6.6}$$

with $SO(3,1)$ index A. The action is now,

$$\begin{aligned}\mathcal{S} =& \frac{1}{g_{ym}^2}\frac{N}{4\pi}\,\eta^2 \int d^4x \int d\Omega\Big\{ -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - 2\partial_\mu(\delta\Phi_i^\dagger)\partial^\mu(\delta\Phi_i) - \frac{i}{2}\,\eta\bar{\Lambda}\gamma^\mu\partial_\mu\Lambda \\ & - \frac{i}{2}\,\eta\bar{\Psi}_i\gamma^\mu\partial_\mu\Psi_i + \eta^2\Big(2(L_iA_\mu)(L_iA^\mu) + i\bar{\Psi}_i\varepsilon_{ijk}\,L_k\Psi_j + 2i\bar{\Psi}_iL_i\Lambda \\ & - \bar{\Psi}_i\Psi_i - 8\delta\Phi_i^\dagger\Big(L^2\,\delta\Phi_i\Big) - 8i\varepsilon_{ijk}\,\delta\Phi_i^\dagger\Big(L_j\,\delta\Phi_k\Big) - 8\delta\Phi_i^\dagger\delta\Phi_i\Big)\Big\}\end{aligned} \tag{6.7}$$

The action remains in the form inherited from the four-dimensional theory, it does not have the canonical form of a six-dimensional theory. Our calculation of the Higgsed $\mathcal{N}=1^*$ mass spectrum suggests that the eigenstates of the Higgsed $\mathcal{N}=1^*$ theory are the eigenstates of the sphere. We now perform a similarity transformation on these $\mathcal{N}=1^*$ fields and expand them in fields on the sphere. Our action will take the canonical form of a six-dimensional field theory. We begin with the scalar potential.

$$V = 8\eta^4 \int d\Omega\,\delta\Phi_i^\dagger\Delta_{ij}\delta\Phi_j \tag{6.8}$$

$$\Delta_{ij} = (L^2+1)\delta_{ij} - i\varepsilon_{ijk}L_k \tag{6.9}$$

We can express the complex scalars as $\delta\Phi_i = \frac{1}{\sqrt{2}}(a_i + ib_i)$ (a_i and b_i real) and define,

$$\mathcal{Y}_i^{\hat{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} a_i \\ ib_i \end{pmatrix} \tag{6.10}$$

where the index $\hat{\alpha}$ labels the two components. The scalar potential is,

$$V = 8\eta^4 \int d\Omega\, \mathcal{Y}^{\dagger}_{i\hat{\alpha}} (\hat{O}_{ij})^{\hat{\alpha}}{}_{\hat{\beta}} \mathcal{Y}_j^{\hat{\beta}}$$

where the matrix $(\hat{O}_{ij})^{\hat{\alpha}}{}_{\hat{\beta}} = \delta^{\hat{\alpha}}{}_{\hat{\beta}} \Delta_{ij}$. We use our knowledge of the eigenstates of the 2-sphere to find the eigenvectors of the operator $\hat{O}_{ij}$. The complete set of eigenvectors of the operator $\hat{O}_{ij}$ are,

$$e_i^{\hat{\alpha}} = v^{\hat{\alpha}} \frac{1}{\sqrt{l(l+1)}}\, iL_i Y_{lm}(\theta,\phi) \tag{6.11a}$$

$$\chi_i^{\hat{\alpha}} = (\sigma_i)^{\hat{\alpha}}{}_{\hat{\beta}} \Omega^{\hat{\beta}}_{q_\pm lm}(\theta,\phi) + \frac{1}{\kappa_\pm} L_i \Omega^{\hat{\alpha}}_{q_\pm lm}(\theta,\phi) \tag{6.11b}$$

where $v^{\hat{\alpha}}$ is an arbitrary 2-component object.

We expand $\mathcal{Y}_i^{\hat{\alpha}}$ in the complete basis of eigenvectors,

$$\mathcal{Y}_i^{\hat{\alpha}} = \mathcal{A}_i^{\hat{\alpha}} + \mathcal{P}_i^{\hat{\alpha}} \tag{6.12}$$

where,

$$\mathcal{A}_i^{\hat{\alpha}} = \sum_{lm} v^{\hat{\alpha}}_{lm} \frac{1}{\sqrt{l(l+1)}}\, iL_i Y_{lm}(\theta,\phi) \tag{6.13a}$$

$$\begin{aligned} \mathcal{P}_i^{\hat{\alpha}} = \sum_{lm} \Bigg\{ & \xi^+_{lm} \left((\sigma_i)^{\hat{\alpha}}{}_{\hat{\beta}} \Omega^{\hat{\beta}}_{q_+lm} + \frac{1}{\kappa_+} L_i \Omega^{\hat{\alpha}}_{q_+lm} \right) \\ & + \xi^-_{lm} \left((\sigma_i)^{\hat{\alpha}}{}_{\hat{\beta}} \Omega^{\hat{\beta}}_{q_-lm} + \frac{1}{\kappa_-} L_i \Omega^{\hat{\alpha}}_{q_-lm} \right) \Bigg\} \end{aligned} \tag{6.13b}$$

$v^{\hat{\alpha}}_{lm}$ is an arbitrary 2-component object for each value $\{l,m\}$ and $\xi^{(\pm)}_{lm}$ is a complex coefficient. We must apply the gauge-fixing condition to our expansion.

$$L_i \delta\Phi_i = \frac{1}{\sqrt{2}} L_i (a_i + ib_i) = 0 \tag{6.14}$$

Consider $v^{\hat{\alpha}}_{lm}$ to be,

$$v^{\hat{\alpha}}_{lm} = \begin{pmatrix} y_{lm} \\ z_{lm} \end{pmatrix}$$

We can identify the fields a_i and b_i,

$$
\begin{aligned}
\frac{1}{\sqrt{2}}\, a_i &= \sum_{lm}\left\{ y_{lm}\frac{1}{\sqrt{l(l+1)}}\, iL_iY_{lm} + \sum_{+,-}\xi^{(\pm)}_{lm}\left(\sigma_i\Omega_{q_\pm lm} + \frac{1}{\kappa_\pm}L_i\Omega_{q_\pm lm}\right)^{\hat{1}}\right\} \\
\frac{1}{\sqrt{2}}\, ib_i &= \sum_{lm}\left\{ z_{lm}\frac{1}{\sqrt{l(l+1)}}\, iL_iY_{lm} + \sum_{+,-}\xi^{(\pm)}_{lm}\left(\sigma_i\Omega_{q_\pm lm} + \frac{1}{\kappa_\pm}L_i\Omega_{q_\pm lm}\right)^{\hat{2}}\right\}
\end{aligned}
$$

The T-spinor term is not constrained by the gauge-fixing as,

$$
L_i\left(\sigma_i\Omega_{q_\pm lm} + \frac{1}{\kappa_\pm}L_i\Omega_{q_\pm lm}\right)^{\hat{\alpha}} = 0 \tag{6.15}
$$

There is no constraint on $\xi^{(\pm)}_{lm}$. Consider the remaining terms,

$$
\begin{aligned}
L_i\delta\Phi_i &= L_i\left(\sum_{lm} y_{lm}\frac{1}{\sqrt{l(l+1)}}\, iL_iY_{lm} + \sum_{lm} z_{lm}\frac{1}{\sqrt{l(l+1)}}\, iL_iY_{lm}\right) \\
&= \sum_{lm}\left(y_{lm} + z_{lm}\right)\frac{1}{\sqrt{l(l+1)}}\, iL^2Y_{lm} = 0
\end{aligned}
$$

The gauge-fixing condition imposes the constraint $y_{lm} = -\, z_{lm}$ which allows us to choose,

$$
v^{\hat{\alpha}}_{lm} = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix} y_{lm} = v^{\hat{\alpha}}\, y_{lm}
$$

hence $v^{\dagger}_{\hat{\alpha}}v^{\hat{\alpha}} = 1$. We can express the expansion $\mathcal{A}^{\hat{\alpha}}_i$ as,

$$
\mathcal{A}^{\hat{\alpha}}_i = v^{\hat{\alpha}}\sum_{lm} y_{lm}\frac{1}{\sqrt{l(l+1)}}\, iL_iY_{lm} \tag{6.16}
$$

The scalar field a_i is real. This imposes a reality condition on the complex coefficient y_{lm}, $a_i = (a_i)^*$ implies,

$$
\begin{aligned}
\sum_{lm} y_{lm} iL_iY_{lm} &= \sum_{lm} y^*_{lm} iL_iY^*_{lm} \\
&= \sum_{lm} y^*_{lm} iL_i(-1)^mY_{l,-m} \\
&= \sum_{lm} y^*_{l,-m}(-1)^{-m} iL_iY_{lm}
\end{aligned}
$$

hence,

$$
y^*_{lm} = (-1)^{-m}y_{l,-m} \tag{6.17}
$$

The eigenvectors of the matrix are orthogonal, therefore the cross-terms are zero. (Suppressing our indices).

$$V = 8\eta^4 \int d\Omega \left\{ \mathcal{A}_i^\dagger \hat{O}_{ij} \mathcal{A}_j + \mathcal{P}_i^\dagger \hat{O}_{ij} \mathcal{P}_j \right\} \tag{6.18}$$

The first term gives,

$$\int d\Omega\, \mathcal{A}_i^\dagger \hat{O}_{ij} \mathcal{A}_j = R^4 \int d\Omega \sum_{lm,l'm'} y_{lm}^\dagger\, y_{l'm'} \frac{1}{\sqrt{l(l+1)l'(l'+1)}} Y_{lm}^\dagger \Delta_{S^2}^2 Y_{l'm'} \tag{6.19}$$

where the Laplacian on the 2-sphere is,

$$\Delta_{S^2} = \frac{1}{\sqrt{g}} \partial_a \left(g^{ab} \sqrt{g}\, \partial_b \right) = \frac{1}{R^2} \left(\cot\theta\, \partial_\theta + \partial_\theta \partial_\theta + \csc^2\ theta\, \partial_\phi \partial_\phi \right) \tag{6.20}$$

In section 2 we expanded vectors on the 2-sphere in the covariant and contravariant vector harmonic, $T_{lm\,a}$ and $T_{lm}{}^{a}$. The definition of these vector harmonics shows us how to define vector fields on the 2-sphere from spherical harmonics,

$$n_\theta(\theta,\phi) = R \sum_{lm} y_{lm} \frac{(-1)}{\sqrt{l(l+1)}} \csc\theta\, \partial_\phi Y_{lm}(\theta,\phi) \tag{6.21a}$$

$$n_\phi(\theta,\phi) = R \sum_{lm} y_{lm} \frac{1}{\sqrt{l(l+1)}} \sin\theta\, \partial_\theta Y_{lm}(\theta,\phi) \tag{6.21b}$$

Clearly the T-vector fields n_a are real. We can write,

$$\begin{aligned} R^2 \sum_{lm} y_{lm} \frac{1}{\sqrt{l(l+1)}} \Delta_{S^2} Y_{lm} &= \sum_{lm} y_{lm} \frac{1}{\sqrt{l(l+1)}} \Big\{ \csc\theta\, \partial_\theta (\sin\theta\, \partial_\theta Y_{lm}) \\ &\qquad + \csc\theta\, \partial_\phi (\csc\theta\, \partial_\phi Y_{lm}) \Big\} \\ &= \frac{1}{R} \left(\csc\theta\, \partial_\theta n_\phi - \csc\theta\, \partial_\phi n_\theta \right) \\ &= \frac{1}{R} \csc\theta\, \mathcal{F}_{\theta\phi} \end{aligned}$$

where $\mathcal{F}_{\theta\phi} = \partial_\theta n_\phi - \partial_\phi n_\theta$. The first term of the scalar potential (6.18) is,

$$\begin{aligned} 8\eta^4 \int d\Omega\, \mathcal{A}_i^\dagger \hat{O}_{ij} \mathcal{A}_j &= 8\eta^4 \frac{1}{R^2} \int d\Omega \csc^2\theta \mathcal{F}_{\theta\phi} \mathcal{F}_{\theta\phi} \\ &= 4\eta^2 \int d\Omega\, \mathcal{F}_{ab} \mathcal{F}^{ab} \end{aligned} \tag{6.22}$$

as $\eta^2 \sim \frac{1}{R^2}$. Consider the second term of the scalar potential,

$$\begin{aligned}
\int d\Omega\, \mathcal{P}_i^\dagger \hat{O}_{ij} \mathcal{P}_j &= \int d\Omega\, \mathcal{P}_i^\dagger \sum_{l'm'} \sum_{+,-} \xi^{(\pm)}_{l'm'}\, \hat{O}_{ij} \left(\sigma_j \Omega_{q'_\pm l'm'} + \frac{1}{\kappa'_\pm} L_j \Omega_{q'_\pm l'm'} \right) \\
&= R^2 \int d\Omega\, \mathcal{P}_i^\dagger \sum_{l'm'} \sum_{+,-} \xi^{(\pm)}_{l'm'} \left(\sigma_i \kappa^2 \Omega_{q'_\pm l'm'} + \frac{1}{\kappa'_\pm} L_i \kappa^2 \Omega_{q'_\pm l'm'} \right) \\
&= R^2 \int d\Omega \sum_{lm,l'm'} \sum_{+,-} \left(\xi^{(\pm)}_{lm} \right)^\dagger \xi^{(\pm)}_{l'm'} \Omega^\dagger_{q_\pm lm} \left(3\kappa^2 + \frac{1}{\kappa'_\pm}\, \sigma_i L_i \kappa^2 \right. \\
&\qquad \left. + \frac{1}{\kappa_\pm}\, \sigma_i L_i \kappa^2 + \frac{1}{\kappa_\pm\, \kappa'_\pm} L^2 \kappa^2 \right) \Omega_{q'_\pm l'm'}
\end{aligned}$$

(Note cross-terms involving $(q_+,\, q'_-)$ and $(q_-,\, q'_+)$ are zero due to the orthogonality of the spherical spinors). Due to the orthogonality condition for the spherical spinors (2.19) we can consider,

$$\begin{aligned}
&\int d\Omega\ \Omega^\dagger_{q_\pm lm} \left(3 + \frac{1}{\kappa'_\pm}\, \sigma_i L_i + \frac{1}{\kappa_\pm}\, \sigma_i L_i + \frac{1}{\kappa_\pm\, \kappa'_\pm} L^2 \right) \Omega_{q'_\pm l'm'} \\
&\qquad = \int d\Omega\ \Omega^\dagger_{q_\pm lm} \left(3 - \frac{2}{\kappa'_\pm} \left(\kappa'_\pm + 1 \right) + \frac{1}{\kappa'^2_\pm} L^2 \right) \Omega_{q'_\pm l'm'} \\
&\qquad = \int d\Omega\ \Omega^\dagger_{q_\pm lm} \left(2 - \frac{1}{\kappa'_\pm} \right) \Omega_{q'_\pm l'm'}
\end{aligned}$$

Therefore if we redefine the complex coefficient,

$$\xi^{(\pm)}_{lm} \rightarrow \sqrt{2 - \frac{1}{\kappa_\pm}}\, \xi^{(\pm)}_{lm}$$

then,

$$\begin{aligned}
8\eta^4 \int d\Omega\, \mathcal{P}_i^\dagger \hat{O}_{ij} \mathcal{P}_j &= 8\eta^4 R^2 \int d\Omega \sum_{lm,l'm'} \sum_{+,-} \left(\xi^{(\pm)}_{lm} \right)^\dagger \xi^{(\pm)}_{l'm'} \Omega^\dagger_{q_\pm lm} \kappa^2 \Omega_{q'_\pm l'm'} \\
&= 8\eta^2 \int d\Omega\, \xi^\dagger_{\hat{\alpha}} \left(\kappa^2 \right)^{\hat{\alpha}}{}_{\hat{\beta}} \xi^{\hat{\beta}}
\end{aligned} \tag{6.23}$$

where,

$$\xi^{\hat{\alpha}}(\theta, \phi) = \sum_{lm} \sum_{+,-} \xi^{(\pm)}_{lm} \Omega^{\hat{\alpha}}_{q_\pm lm}(\theta, \phi) \tag{6.24}$$

is a T-spinor field. The scalar potential is,

$$V = 8\eta^2 \int d\Omega \left\{ \frac{1}{2} \mathcal{F}_{ab} \mathcal{F}^{ab} + \xi^\dagger_{\hat{\alpha}} (\kappa^2)^{\hat{\alpha}}{}_{\hat{\beta}} \xi^{\hat{\beta}} \right\} \tag{6.25}$$

We treat the fermionic part in a similar way.

$$\mathcal{L}_{Fint} = \eta^4 \int d\Omega \left\{ i\bar{\Psi}_i \varepsilon_{ijk} L_k \Psi_j + 2i\bar{\Psi}_i L_i \Lambda - \bar{\Psi}_i \Psi_i \right\} \tag{6.26}$$

Inspired by the SUSY tansformation of $\mathcal{Y}_i^{\hat{\alpha}}$ we define the following 2-component objects,

$$\mathcal{X}_{iA}^{\hat{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Psi_{iA} \\ i\left(\gamma_5\right)_A{}^B \Psi_{iB} \end{pmatrix} \qquad \mathcal{Z}_A^{\hat{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \Lambda_A \\ i\left(\gamma_5\right)_A{}^B \Lambda_B \end{pmatrix}$$

The interacting fermionic part is now,

$$\mathcal{L}_{Fint} = \eta^4 \int d\Omega \left\{ \bar{\mathcal{X}}_{i\hat{\alpha}}^A (\hat{\Delta}_{ij})^{\hat{\alpha}B}_{\hat{\beta}A} \mathcal{X}_{jB}^{\hat{\beta}} + 2i\, \bar{\mathcal{X}}_{i\hat{\alpha}}^A \left(\delta^{\hat{\alpha}}{}_{\hat{\beta}} \delta_A{}^B L_i \right) \mathcal{Z}_B^{\hat{\beta}} \right\} \tag{6.27}$$

with $\hat{\Delta}_{ij} = \delta^{\hat{\alpha}}{}_{\hat{\beta}} \delta_A{}^B \left(i\varepsilon_{ijk} L_k - \delta_{ij} \right)$. We expand the fermions of the chiral multiplets in the complete basis of eigenvectors of $\hat{O}_{ij}$,

$$\mathcal{X}_{iA}^{\hat{\alpha}} = \mathcal{B}_{iA}^{\hat{\alpha}} + \mathcal{R}_{iA}^{\hat{\alpha}} \tag{6.28}$$

with,

$$\mathcal{B}_{iA}^{\hat{\alpha}} = \sum_{lm} u_{lmA}^{\hat{\alpha}} \frac{1}{\sqrt{l(l+1)}}\, iL_i\, Y_{lm}(\theta,\phi) \tag{6.29a}$$

$$\mathcal{R}_{iA}^{\hat{\alpha}} = \sum_{lm} \sum_{+,-} \zeta_{lmA}^{(\pm)} \left((\sigma_i)^{\hat{\alpha}}{}_{\hat{\beta}} \Omega^{\hat{\beta}}_{q\pm lm}(\theta,\phi) + \frac{1}{\kappa_\pm} L_i \Omega^{\hat{\alpha}}_{q\pm lm}(\theta,\phi) \right) \tag{6.29b}$$

where $\zeta_{lmA}^{(\pm)}$ is a $SO(3,1)$ spinor coefficient. The coefficient $u_{lmA}^{\hat{\alpha}}$ is a $SO(3,1)$ spinor for each $\{l, m\}$ and is a 2-component object like its bosonic counterpart $v_{lm}^{\hat{\alpha}}$.

$$u_{lmA}^{\hat{\alpha}} = \frac{1}{\sqrt{2}} \begin{pmatrix} \delta_A{}^B \\ i\left(\gamma_5\right)_A{}^B \end{pmatrix} u_{lmB}$$

Consider the terms,

$$\eta^4 \int d\Omega \left\{ i\bar{\Psi}_i \varepsilon_{ijk} L_k \Psi_j - \bar{\Psi}_i \Psi_i \right\} = \eta^4 \int d\Omega \Big\{ \bar{\mathcal{B}}_i \hat{\Delta}_{ij} \mathcal{B}_j + \bar{\mathcal{B}}_i \hat{\Delta}_{ij} \mathcal{R}_j + \bar{\mathcal{R}}_i \hat{\Delta}_{ij} \mathcal{B}_j + \bar{\mathcal{R}}_i \hat{\Delta}_{ij} \mathcal{R}_j \Big\}$$

The only non-zero terms are,

$$\eta^4 \int d\Omega\, \bar{\mathcal{R}}_i \hat{\Delta}_{ij} \mathcal{R}_j = \eta^3 \int d\Omega\, \bar{\zeta}^A_{\hat{\alpha}}(\theta,\phi)\, \kappa^{\hat{\alpha}}{}_{\hat{\beta}}\, \zeta^{\hat{\beta}}_A(\theta,\phi) \tag{6.30}$$

where $\zeta(\theta,\phi)$ is a fermionic T-spinor,

$$\zeta_A^{\hat{\alpha}}(\theta,\phi) = \sum_{lm}\sum_{+,-}\zeta_{lmA}^{(\pm)}\Omega_{q_\pm lm}^{\hat{\alpha}} \tag{6.31}$$

The remaining term of the interacting fermionic part is,

$$2\eta^2\int d\Omega\, i\bar{\Psi}_i L_i\Lambda = 2\eta^2\int d\Omega\left\{i\bar{\mathcal{B}}_{i\hat{\alpha}}^A L_i\mathcal{Z}_A^{\hat{\alpha}} + i\bar{\mathcal{R}}_{i\hat{\alpha}}^A L_i\mathcal{Z}_A^{\hat{\alpha}}\right\} \tag{6.32}$$

which contains the gaugino Λ. The 2-component object $\mathit{\Lambda}_A$ is expanded in spherical harmonics,

$$\mathcal{Z}_A^{\hat{\alpha}}(\theta,\phi) = \sum_{lm}\mathcal{Z}_{lmA}^{\hat{\alpha}}Y_{lm}(\theta,\phi) \tag{6.33}$$

where $\mathcal{Z}_{lmA}^{\hat{\alpha}}$ is a $SO(3,1)$ spinor coefficient and an arbitrary 2-component object like $u_{lmA}^{\hat{\alpha}}$.

$$\mathcal{Z}_{lmA}^{\hat{\alpha}} = \frac{1}{\sqrt{2}}\begin{pmatrix} delta_A{}^B \\ i\left(\gamma_5\right)_A{}^B\end{pmatrix}\Lambda_{lmB}$$

The second term of (6.31) vanishes, but the first term is,

$$\int d\Omega\; i\bar{\mathcal{B}}_{iA}^{\hat{\alpha}}L_i\mathcal{Z}_{\hat{\alpha}}^A = R\int d\Omega\,\frac{1}{\sqrt{g}}\,\bar{\mathcal{G}}_{\theta\phi}^A\Lambda_A \tag{6.34}$$

where $\mathcal{G}_{ab} = \partial_a g_b - \partial_b g_a$ is the 'field tensor' of the fermionic T-vector $g_a(\theta,\phi)$ in analogy with the bosons,

$$\begin{aligned}
g_{\theta A}(\theta,\phi) &= R\sum_{lm}u_{lmA}\frac{(-1)}{\sqrt{l(l+1)}}\csc\theta\,\partial_\phi Y_{lm}(\theta,\phi) && (6.35\text{a})\\
g_{\phi A}(\theta,\phi) &= R\sum_{lm}u_{lmA}\frac{1}{\sqrt{l(l+1)}}\sin\theta\,\partial_\theta Y_{lm}(\theta,\phi) && (6.35\text{b})\\
\frac{1}{\sqrt{g}}\,\mathcal{G}_{\theta\phi A} &= R\sum_{lm}u_{lmA}\frac{1}{\sqrt{l(l+1)}}\,\Delta_{S^2}Y_{lm}(\theta,\phi) && (6.35\text{c})
\end{aligned}$$

and,

$$\Lambda_A(\theta,\phi) = \sum_{lm}\Lambda_{lmA}Y_{lm}(\theta,\phi)$$

Therefore,

$$2\eta^4\int d\Omega\, i\bar{\mathcal{B}}_i L_i\mathcal{Z} = \eta^3\int d\Omega\,\frac{1}{\sqrt{g}}\,\varepsilon^{ab}\bar{\mathcal{G}}_{ab}\Lambda \tag{6.36}$$

where $\varepsilon^{\theta\phi} = 1$. The fermionic part has become,

$$\mathcal{L}_{Fint} = \eta^3 \int d\Omega \Big\{ \bar{\zeta}_{\hat{\alpha}}\, \kappa^{\hat{\alpha}}{}_{\hat{\beta}}\, \zeta^{\hat{\beta}} + \frac{1}{\sqrt{g}}\, \varepsilon^{ab} \bar{\mathcal{G}}_{ab} \Lambda \Big\} \tag{6.37}$$

Consider the terms originating from the $\mathcal{N} = 1^*$ bosonic kinetic terms,

$$\begin{aligned}\mathcal{L}_{Bkin} = \frac{1}{g_{ym}^2} \frac{N}{4\pi R^2}\, \eta^2 \int d\Omega \Big\{ &- \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 2\partial_\mu(\delta\Phi_i^\dagger)\partial^\mu(\delta\Phi_i) \\ &+ 2\eta^2 (L_i A_\mu)(L_i A^\mu) \Big\}\end{aligned} \tag{6.38}$$

The gauge boson is expanded in spherical harmonics,

$$A_\mu(\theta, \phi) = \sum_{lm} A_{(\mu)lm} Y_{lm}(\theta, \phi) \tag{6.39}$$

where $A_{(\mu)lm}$ is a complex coefficient. The first term in (6.38) is trivial. The third term is clearly,

$$2\eta^2 \int d\Omega (L_i A_\mu)(L_i A^\mu) = 2 \int d\Omega\; A_\mu \Delta_{S^2} A^\mu \tag{6.40}$$

The second term is,

$$\begin{aligned} 2\eta^2 \int d\Omega\, \partial_\mu(\delta\Phi_i^\dagger)\partial^\mu(\delta\Phi_i) &= 2\eta^2 \int d\Omega\; \partial_\mu \mathcal{Y}_i^\dagger \partial^\mu \mathcal{Y}_i \\ &= 2\eta^2 \int d\Omega\; \Big(\partial_\mu \mathcal{A}_i^\dagger \partial^\mu \mathcal{A}_i + \partial_\mu \mathcal{P}_i^\dagger \partial^\mu \mathcal{P}_i \Big) \end{aligned}$$

We ignore the cross-terms due to the orthogonality of the eigenvectors,

$$\begin{aligned} &2\eta^2 \int d\Omega\, \partial_\mu \mathcal{A}_i^\dagger \partial^\mu \mathcal{A}_i \\ &\qquad = 2\eta^2 \int d\Omega \sum_{lm} \sum_{l'm'} \partial_\mu y_{lm}^\dagger\, \partial^\mu y_{l'm'} \frac{1}{\sqrt{l(l+1)l'(l'+1)}}\, Y_{lm}^\dagger L^2 Y_{l'm'} \end{aligned}$$

We can use equation (2.6) to re-express,

$$\sum_{lm} y_{lm} \frac{1}{\sqrt{l(l+1)}}\, iL_i\, Y_{lm} \;=\; \sum_{lm} y_{lm} \frac{1}{\sqrt{l(l+1)}}\, k_i^a \partial_a Y_{lm} = \frac{1}{R} \frac{1}{\sqrt{g}}\, k_i^a g_{ab}\, \varepsilon^{bc} n_c$$

Therefore,

$$\begin{aligned} 2\eta^2 \int d\Omega\, \partial_\mu \mathcal{A}_i^\dagger \partial^\mu \mathcal{A}_i &= 2\eta^2 \frac{1}{R^2} \int d\Omega\, \partial_\mu \left(\frac{1}{\sqrt{g}}\, k_i^a g_{ab}\, \varepsilon^{bc} n_c \right) \partial^\mu \left(\frac{1}{\sqrt{g}}\, k_i^d g_{de}\, \varepsilon^{ef} n_f \right) \\ &= 2\eta^2 \frac{1}{R^2} \int d\Omega\, \frac{1}{g}\, \varepsilon^{bc} g_{be}\, \varepsilon^{ef} \partial_\mu n_c\, \partial^\mu n_f \\ &= 2\eta^2 \int d\Omega\, \partial_\mu n_a \partial^\mu n^a \end{aligned} \tag{6.41}$$

The final term of the bosonic kinetic term is,

$$\begin{aligned}\int d\Omega\, \partial_\mu \mathcal{P}_i^\dagger \partial^\mu \mathcal{P}_i &= \int d\Omega \sum_{lm,l'm'} \sum_{+,-} \partial_\mu (\xi_{lm}^{(\pm)})^\dagger \partial^\mu \xi_{l'm'}^{(\pm)} \left(\Omega_{q_\pm lm}^\dagger \sigma_i + \frac{1}{\kappa_\pm} \Omega_{q_\pm lm}^\dagger L_i \right) \\ &\quad \times \left(\sigma_i \Omega_{q'_\pm l'm'} + \frac{1}{\kappa'_\pm} L_i \Omega_{q'_\pm l'm'} \right) \\ &\to \int d\Omega\, \partial_\mu \xi_{\hat{\alpha}}^\dagger(\theta,\phi) \partial^\mu \xi^{\hat{\alpha}}(\theta,\phi)\end{aligned} \tag{6.42}$$

The kinetic term of the gauginos is trivial. The kinetic term of the chiral spinors is,

$$\begin{aligned}\eta^3 \int d\Omega\, \bar{\Psi}_i^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \Psi_{iB} &= \int d\Omega\, \bar{\mathcal{X}}_{i\hat{\alpha}}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \mathcal{X}_{iB}^{\hat{\alpha}} \\ &= \eta^3 \int d\Omega \Big\{ \bar{\mathcal{B}}_{i\hat{\alpha}}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \mathcal{B}_{iB}^{\hat{\alpha}} + \bar{\mathcal{R}}_{i\hat{\alpha}}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \mathcal{R}_{iB}^{\hat{\alpha}} \Big\}\end{aligned} \tag{6.43}$$

Due to the orthogonality of the eigenvectors we ignore the cross-terms. Consider,

$$\begin{aligned}\eta^3 \int d\Omega\, \bar{\mathcal{B}}_{i\hat{\alpha}}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \mathcal{B}_{iB}^{\hat{\alpha}} &= \eta^3 \int d\Omega \sum_{lm} \sum_{l'm'} \left(\bar{u}_{lm}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu u_{l'm'B} \right) \\ &\quad \times \frac{1}{\sqrt{l(l+1)l'(l'+1)}} Y_{lm}^\dagger L^2 Y_{l'm'}\end{aligned}$$

Now in analogy with the bosons,

$$\sum_{lm} u_{lmA} \frac{1}{\sqrt{l(l+1)}} iL_i Y_{lm} = \frac{1}{R} \frac{1}{\sqrt{g}}\, k_i^a g_{ab} \varepsilon^{bc} g_{cA} \tag{6.44}$$

Therefore,

$$\begin{aligned}\eta^3 \int d\Omega\, \bar{\mathcal{B}}_{i\hat{\alpha}}^A \left(\gamma^\mu\right)_A{}^B \partial_\mu \mathcal{B}_{iB}^{\hat{\alpha}} &= \eta^3 \int d\Omega\, \frac{1}{R^2} \frac{1}{g}\, k_i^a g_{ab} \varepsilon^{bc} k_i^d g_{de} \varepsilon^{ef} \bar{g}_c^A \left(\gamma^\mu\right)_A{}^B \partial_\mu g_{fB} \\ &= \eta^3 \int d\Omega\, \bar{g}_a^A \left(\gamma^\mu\right)_A{}^B \partial_\mu g_B^a\end{aligned} \tag{6.45}$$

Consider the final term,

$$
\begin{aligned}
\eta^3 \int d\Omega \, \bar{\mathcal{R}}^A_{i\hat{\alpha}} \left(\gamma^\mu\right)_A{}^B \partial_\mu \mathcal{R}^{\hat{\alpha}}_{iB} \\
&= \eta^3 \int d\Omega \sum_{lm} \sum_{l'm'} \sum_{+,-} \left(\bar{\zeta}^{(\pm)A}_{lm} \left(\gamma^\mu\right)_A{}^B \partial_\mu \zeta^{(\pm)}_{l'm'B} \right) \\
&\quad \times \left(\Omega^\dagger_{q_\pm lm} \sigma_i + \frac{1}{\kappa_\pm} \Omega^\dagger_{q_\pm lm} L_i \right)_{\hat{\alpha}} \left(\sigma_i \Omega_{q'_\pm l'm'} + \frac{1}{\kappa'_\pm} L_i \Omega_{q'_\pm l'm'} \right)^{\hat{\alpha}} \\
&\to \eta^3 \int d\Omega \, \bar{\zeta}^A_{\hat{\alpha}} \left(\gamma^\mu\right)_A{}^B \partial_\mu \zeta^{\hat{\alpha}}_B
\end{aligned}
\tag{6.46}
$$

Summarizing the above calculations, the resulting six-dimensional effective action (contracting all spinor indices),

$$
\begin{aligned}
\mathcal{S} = \frac{1}{g^2_{ym}} \frac{N}{4\pi} \eta^2 \int d^4x \int d\Omega \Big\{ & - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{i}{2} \eta \bar{\Lambda} \gamma^\mu \partial_\mu \Lambda - \frac{i}{2} \eta \bar{g}_a \gamma^\mu \partial_\mu g^a \\
& - \frac{i}{2} \eta \bar{\zeta} \gamma^\mu \partial_\mu \zeta - 2 \partial_\mu n_a \partial^\mu n^a - 2 \partial_\mu \xi^\dagger \partial^\mu \xi + 2 A_\mu \Delta_{S^2} A^\mu \\
& - 4 \mathcal{F}_{ab} \mathcal{F}^{ab} - 8 \xi^\dagger \kappa^2 \xi + \eta \, \bar{\zeta} \, \kappa \, \zeta + \frac{1}{\sqrt{g}} \eta \, \varepsilon^{ab} \bar{\mathcal{G}}_{ab} \Lambda \Big\}
\end{aligned}
\tag{6.47}
$$

The T-spinors are all expressed in terms of the cartesian basis of spherical spinors, however in our calculation of the MN spectrum we used spherical spinors in the spherical basis, therefore we will express the action in this basis.

$$
\xi = V^\dagger \Xi \qquad \qquad \zeta = V^\dagger \Upsilon \tag{6.48}
$$

Which gives the action,

$$
\begin{aligned}
\mathcal{S} = \frac{1}{g^2_{ym}} \frac{N}{4\pi R^2} \eta^2 \int d^4x \int R^2 d\Omega \Big\{ & - \frac{1}{4} F_{\mu\nu} F^{\mu\nu} - \frac{i}{2} \eta \, \bar{\Lambda} \gamma^\mu \partial_\mu \Lambda - \frac{i}{2} \eta \, \bar{g}_a \gamma^\mu \partial_\mu g^a \\
& - \frac{i}{2} \eta \, \bar{\Upsilon} \gamma^\mu \partial_\mu \Upsilon - 2 \partial_\mu n_a \partial^\mu n^a - 2 \partial_\mu \Xi^\dagger \partial^\mu \Xi + 2 A_\mu \Delta_{S^2} A^\mu - 4 \mathcal{F}_{ab} \mathcal{F}^{ab} \\
& - 8 \, \Xi^\dagger (-i \hat{\nabla}_{S^2})^2 \, \Xi + \eta \, \bar{\Upsilon} \hat{\gamma}_3 (-i \hat{\nabla}_{S^2}) \Upsilon + \frac{1}{\sqrt{g}} \eta \, \varepsilon^{ab} \bar{\mathcal{G}}_{ab} \Lambda \Big\}
\end{aligned}
\tag{6.49}
$$

The above action has the canonical form of a six-dimensional theory and the calculation of Sections 5 implies that it has the correct spectrum. The action contains no interaction terms as one would expect in a $U(1)$ gauge

theory with adjoint matter and there are also no explicit mass terms. The six-dimensional theory has intrinsically massless fields propagating along the flat four dimensional spacetime and along the compact 2-manifold. It is the compactness of the 2-manifold which gives mass to the theory. This is to be expected as the MN compactification is a twisted compactification of the massless $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory. The fields are summarized below.

Fields	T-Spin	Spin
A_μ	T-scalar	
Ξ	T-spinor	Bosons
n_a	T-vector	
Λ	T-scalar	
Υ	T-spinor	Fermions
g_a	T-vector	

We can see how the twisted compactification of Maldacena and Núñez is realized in the classical action. The bosonic T-scalars, fermionic T-spinors and bosonic T-vectors are all realized as in an untwisted field theory. The kinetic term of the bosonic T-spinors on the 2-sphere is realized as the square of the Dirac operator. This makes perfect sense. The kinetic term of a boson is quadratic in derivatives whilst a fermion is linear in derivatives. It is not possible to remove a derivative from the action, therefore the bosonic T-spinors must be quadratic in derivatives. In a standard field theory the spinors have a Dirac operator, linear in derivatives, so it is sensible for the kinetic term of the bosonic T-spinor to be the square of the Dirac operator on the sphere. We find the fermionic T-scalars and fermionic T-vectors have a coupled kinetic term. We can think of this term as a combined square root of a scalar Laplacian and a Maxwell term. This makes sense for the same reason as the bosonic T-spinor. The fields are fermions and hence linear in derivatives, so the kinetic term for the fermionic T-scalars and T-vectors must be a "square root" of the canonical kinetic term for scalars and vectors, respectively. The origin of the "coupled kinetic term" can be seen in the $\mathcal{N} = (1,1)$ action (A.12). The fields $\lambda^A_{\underline{\alpha}}$ are in the **4** representation of $SU(4)$. It is this $\mathcal{N} = (1,1)$ spinor which forms the fermionic T-scalars and fermionic T-vectors of the Maldacena-Núñez compactification. Therefore the term,

$$\lambda^{A\underline{\alpha}}\bar{\Sigma}^i_{AB}\partial_i\lambda^B_{\underline{\alpha}}$$

from the $\mathcal{N} = (1,1)$ action will upon twisted compactification give rise to the term coupling the fermionic T-scalars and fermionic T-vectors. Futhermore,

the Kaluza-Klein spectrum of the MN compactifcation suggests a coupling between the fermionic T-scalars and T-vectors. In order to organise the MN fields into $\mathcal{N} = 1$ multiplets we had to combine the T-scalars with the T-vectors to obtain massive vector multiplets.

Finally we can also see that the effective action calculated above is identical to the bosonic action (3.13) derived from the $\mathcal{N} = (1,1)$ SUSY Yang-Mills action. In particular, with appropriate rescaling of the fields[7], fixing the gauge $\partial_a n^a = 0$ and the integration by parts of the kinetic term of the $U(1)$ gauge field, (3.13) is identical to the bosonic part of (6.47), with six-dimensional coupling $g_6^2 = 4\pi R^2 g_{ym}^2/N$. As the $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory is supersymmetric this is sufficient evidence to show that the Maldacena-Núñez compactification is equivalent to the Higgsed $\mathcal{N} = 1^*$ SUSY Yang-Mills theory in the limit $N \to \infty$.

The work of ND is supported by a PPARC senior fellowship. RPA would like to thank Tim Hollowood and Carlos Núñez for their help, support and suggestions.

A Clifford Algebras

Our notation generally follows that of Wess and Bagger [25], with metric convention $(-,+,+,+,\dots)$. For the four-dimensional theories we use the $SO(3,1)$ Clifford algebra of Wess and Bagger,

$$\gamma^\mu = \begin{pmatrix} 0 & \sigma^\mu \\ \bar{\sigma}^\mu & 0 \end{pmatrix} \tag{A.1}$$

The $SO(3,1)$ chirality matrix is,

$$\gamma^5 = \gamma^0\gamma^1\gamma^2\gamma^3 = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \tag{A.2}$$

We derive the $\mathcal{N} = (1,1)$ SUSY Yang-Mills theory in a six-dimensional spacetime from the $\mathcal{N} = 1$ SUSY Yang-Mills theory in a ten-dimensional spacetime, via Trivial Dimensional Reduction. The action for $\mathcal{N} = 1$ SUSY Yang-Mills in ten spacetime dimensions is,

$$\mathcal{S} = \frac{1}{g^2}\int d^{10}x \left(-\frac{1}{4}F_{MN}F^{MN} - \frac{i}{2}\bar{\Psi}\Gamma^M\partial_M\Psi\right) \tag{A.3}$$

[7]This including the absorbing of the mass parameter η to return the fields to the correct mass dimension.

$F_{MN} = \partial_M A_N - \partial_N A_M$, where A_M is a ten-dimensional gauge field and Ψ is a 32-component spinor of $SO(9,1)$, $M = 0,1,\ldots,9$. Trivial dimensional reduction assumes the ten dimensional fields in the above action are dependent on only the first six dimensions x^i, where $i = 0,1,\ldots,5$.

$$F_{MN}F^{MN} \;=\; F_{ij}F^{ij} + 2\,\partial_i\phi_m\,\partial^i\phi^m$$

where $\phi_m = A_m$, $m = 1,2,3,4$. To calculate the spinor contribution we must decompose the Majorana-Weyl spinor of $SO(9,1)$ in to a representation of $SO(5,1)\times SO(4)$.

$$\mathbf{16} \to (\mathbf{4},\mathbf{2},\mathbf{1}) \oplus (\bar{\mathbf{4}},\mathbf{1},\mathbf{2}) \tag{A.4}$$

This decomposition is obtained by the following Clifford algebra decomposition of $SO(9,1)$.

$$\Gamma^M = \left\{\tilde{\Gamma}^i \otimes \tilde{\gamma}^5, \mathbb{1}_8 \otimes \tilde{\gamma}^m\right\} \tag{A.5}$$

where $\tilde{\Gamma}^i$ is the $SO(5,1)$ Clifford algebra and $\tilde{\gamma}^m$ is the $SO(4)$ Clifford algebra. The algebra for $SO(5,1)$ follows from [26],

$$\tilde{\Gamma}^i = \begin{pmatrix} 0 & \Sigma^i \\ \bar{\Sigma}^i & 0 \end{pmatrix} \tag{A.6}$$

where,

$$\Sigma^i \;=\; \left(-i\eta^3, i\bar{\eta}^3, \eta^2, i\bar{\eta}^2, \eta^1, i\bar{\eta}^1\right) \tag{A.7a}$$
$$\bar{\Sigma}^i \;=\; \left(i\eta^3, i\bar{\eta}^3, -\eta^2, i\bar{\eta}^2, -\eta^1, i\bar{\eta}^1\right) \tag{A.7b}$$

where η^c and $\bar{\eta}^c$ are the t'Hooft eta symbols. The algebra for $SO(4)$ is [26],

$$\tilde{\gamma}^m = \begin{pmatrix} 0 & \tau^m \\ \bar{\tau}^m & 0 \end{pmatrix} \tag{A.8}$$

where $\tau_m = (\vec{\sigma}, -i\mathbb{1})$ and $\bar{\tau}_m = (\vec{\sigma}, i\mathbb{1})$.

From the chirality condition the spinor Ψ decomposes into,

$$\Psi = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix} \lambda^A_{\underline{\alpha}} + \begin{pmatrix} 0 \\ 1 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix} \bar{\lambda}^{\dot{\underline{\alpha}}}_A \tag{A.9}$$

where $A = 1,2,3,4$ is an $SO(5,1)$ spinor index and $\underline{\alpha}, \dot{\underline{\alpha}} = 1,2$ are $SU(2)_A \times SU(2)_B \sim SO(4)$ spinor indices. From the Majorana condition the Weyl spinors have the following hermitian conjugates.

$$\left(\lambda^A_{\underline{\alpha}}\right)^\dagger \;=\; \bar{\Sigma}^0_{AB}\lambda^{B\underline{\alpha}} \tag{A.10a}$$
$$\left(\bar{\lambda}^{\dot{\underline{\alpha}}}_A\right)^\dagger \;=\; \Sigma^{0AB}\bar{\lambda}_{B\dot{\underline{\alpha}}} \tag{A.10b}$$

The fermionic term is,

$$i\bar{\Psi}\Gamma^M\partial_M\Psi \;=\; i\lambda^{A\underline{\alpha}}\bar{\Sigma}^i_{AB}\partial_i\lambda^B_{\underline{\alpha}} + i\bar{\lambda}_{A\underline{\dot{\alpha}}}\Sigma^{iAB}\partial_i\bar{\lambda}^{\underline{\dot{\alpha}}}_B \tag{A.11}$$

and the full $\mathcal{N}=(1,1)$ SUSY Yang-Mills actions is,

$$\begin{aligned}\mathcal{S} \;=\; \frac{1}{g_6^2}\int d^6x\,\bigg(&-\frac{1}{4}F_{ij}F^{ij} - \frac{1}{2}\,\partial_i\phi_m\,\partial^i\phi^m\bigg) \\ &-\frac{i}{2}\lambda^{A\underline{\alpha}}\bar{\Sigma}^i_{AB}\partial_i\lambda^B_{\underline{\alpha}} - \frac{i}{2}\bar{\lambda}_{A\underline{\dot{\alpha}}}\Sigma^{iAB}\partial_i\bar{\lambda}^{\underline{\dot{\alpha}}}_B\bigg)\end{aligned} \tag{A.12}$$

The $SO(4)$ Clifford algebra has the following identities,

$$\mathrm{Tr}\,\tau^m\bar{\tau}^n \;=\; 2\,\delta^{mn} \tag{A.13a}$$

$$(\tau^m)_{\underline{\alpha}\underline{\dot{\alpha}}}(\bar{\tau}_m)^{\underline{\dot{\beta}\beta}} \;=\; 2\,\delta_{\underline{\alpha}}{}^{\underline{\beta}}\delta_{\underline{\dot{\alpha}}}{}^{\underline{\dot{\beta}}} \tag{A.13b}$$

$$(\bar{\tau}^m)_{\underline{\alpha}\underline{\dot{\alpha}}}(\bar{\tau}_m)^{\underline{\dot{\beta}\beta}} \;=\; -2\,\varepsilon^{\underline{\alpha\beta}}\varepsilon^{\underline{\dot{\alpha}\dot{\beta}}} \tag{A.13c}$$

The $SO(4)$ bispinor has the hermitian conjugate,

$$\begin{aligned}\left(v_{\underline{\alpha}}{}^{\underline{\dot{\alpha}}}\right)^\dagger &= (\bar{\lambda}^{\underline{\dot{\alpha}}}_A)^\dagger(\lambda^A_{\underline{\alpha}})^\dagger \\ &= \Sigma^{0AB}\bar{\lambda}_{B\underline{\dot{\alpha}}}\bar{\Sigma}^0_{AC}\lambda^{C\underline{\alpha}} \\ &= \bar{\lambda}_{A\underline{\dot{\alpha}}}\lambda^{A\underline{\alpha}} \\ &= -\lambda^{A\underline{\alpha}}\bar{\lambda}_{A\underline{\dot{\alpha}}} \\ &= -v^{\underline{\alpha}}{}_{\underline{\dot{\alpha}}}\end{aligned}$$

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